Laplasova transformacija

$$L\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

$$L^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} F(s)\cdot e^{st}\cdot ds, \quad t > 0$$

- $L\{.\}$ – Laplasova transformacija
- $L^{-1}\{.\}$ – Inverzna Laplasova transformacija
- s – kompleksna učestanost (komp. prom. Laplasove trans.)
- $F(s)$ – kompleksan lik funkcije $f(t)$
- $f(t)$ – original funkcije $F(s)$

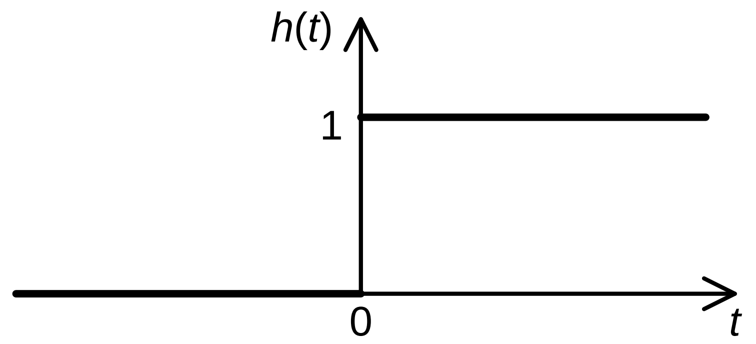
Osobine Laplasove transformacije

1. Teorema linearnosti $L\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(s) + a_2 F_2(s)$
2. Čisto vremensko kašnjenje $L\{ f(t-\tau)\} = e^{-s\tau} F(s)$
3. Pomeranje kompleksnog lika $L\{e^{-at} f(t)\} = F(s+a)$
4. Konvolucija originala $L(f_1(t) * f_2(t)\} = F_1(s) \cdot F_2(s)$
5. Teorema o izvodu originala $L\{\frac{d}{dt} f(t)\} = sF(s) - f(0_-)$
6. Teorema o integralu originala $L\{\int_0^t f(t)dt\} = \frac{F(s)}{s}$
7. Teorema o izvodu kompleksnog lika $L\{ f^n(t)\} = (-1)^n \frac{d^n F(S)}{ds^n}$
8. Teorema o promeni vremenske skale $L\{ f(\frac{t}{a})\} = aF(as)$
9. Prva granična teorema $\lim_{t \to 0_+} f(t) = \lim_{s \to \infty} sF(s)$
10. Druga granična teorema $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

Tablica Laplasove transformacije

$f(t)$	$F(s)$
$\delta(t)$	1
$t^n h(t)$	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$e^{-\alpha t}\cos(\omega t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$e^{-\alpha t}\sin(\omega t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$

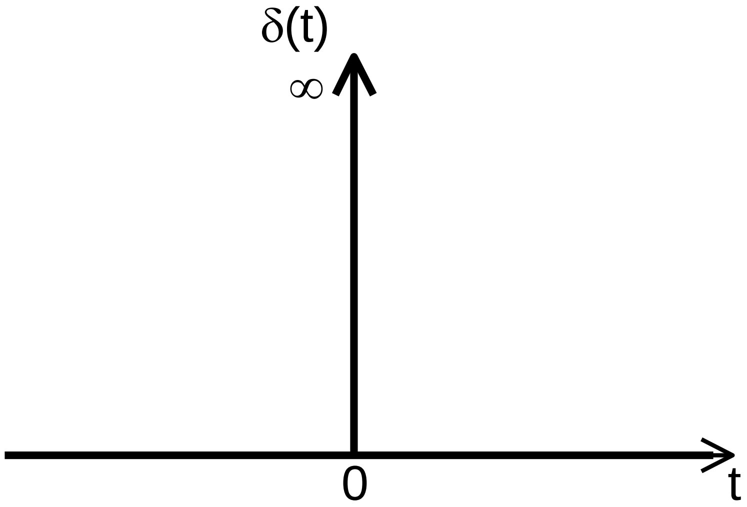
Hevisajdov signal



$$h(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

$$F(s) = L\{f(t)\} = \int_{0+}^{\infty} f(t)e^{-st}dt = \int_{0+}^{\infty} 1 \cdot e^{-st}dt = -\left.\frac{e^{-st}}{s}\right|_{0}^{\infty} = \frac{1}{s}$$

Dirakov impuls

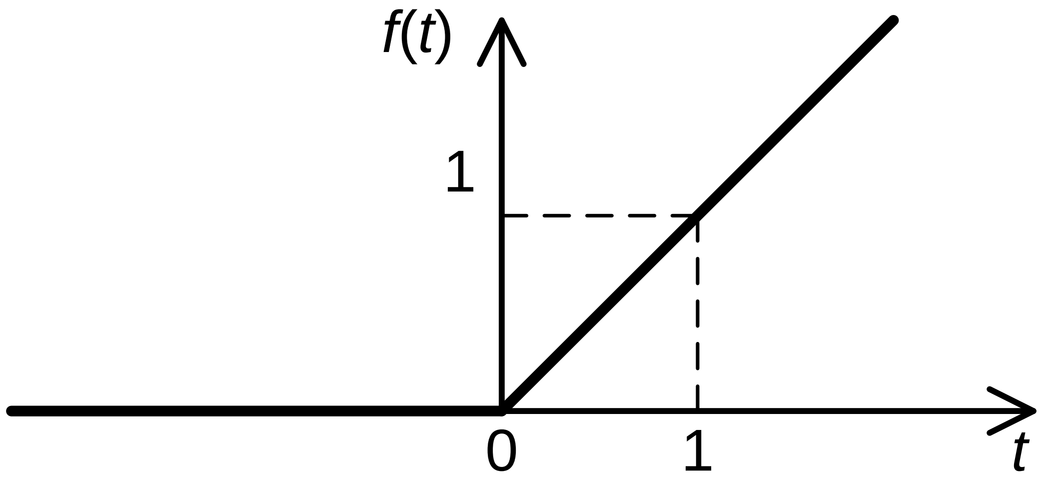


$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$F(s) = L\{\frac{dh(t)}{dt}\} = sL\{h(t)\} - h(0_{-}) = s \cdot \frac{1}{s} - 0 = 1$$

Jedinični nagibni signal

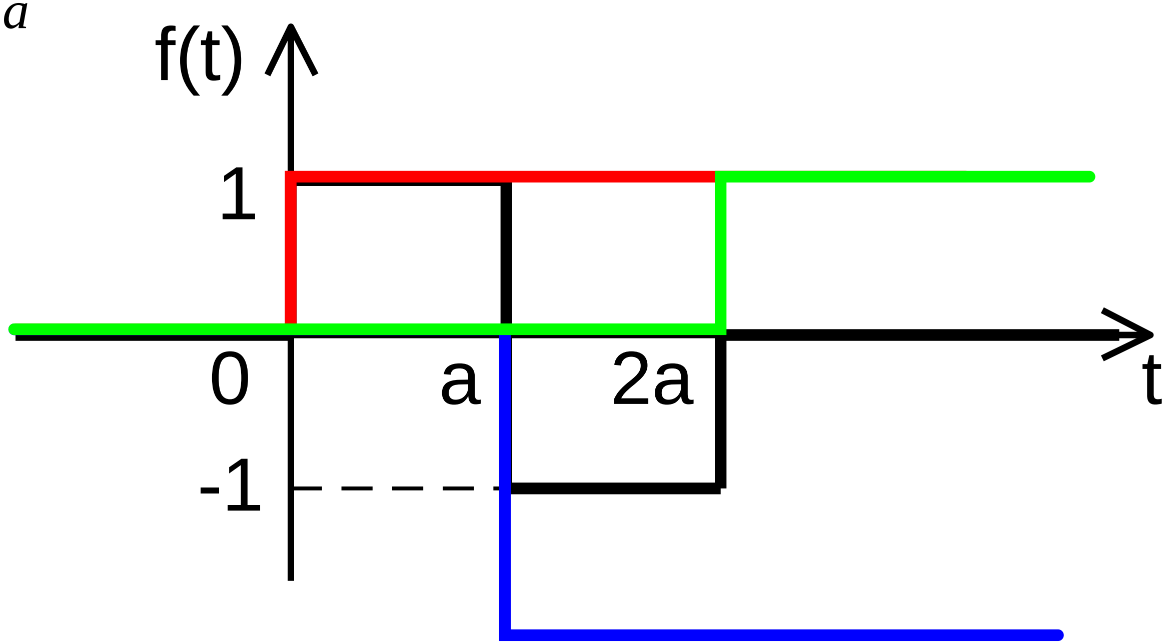


$$h(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$

$$F(s) = L\{f(t)\} = L\{t \cdot h(t)\} = \int_0^\infty t \cdot e^{-st} dt = -\frac{t \cdot e^{-st}}{s}\bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} dt = \frac{1}{s^2}$$

Primer – složen signal 1

$$h(t)=\begin{cases}0, & t<0\\ 1, & 0\le t<a\\ -1, & a\le t<2a\\ 0, & t\ge a\end{cases}$$

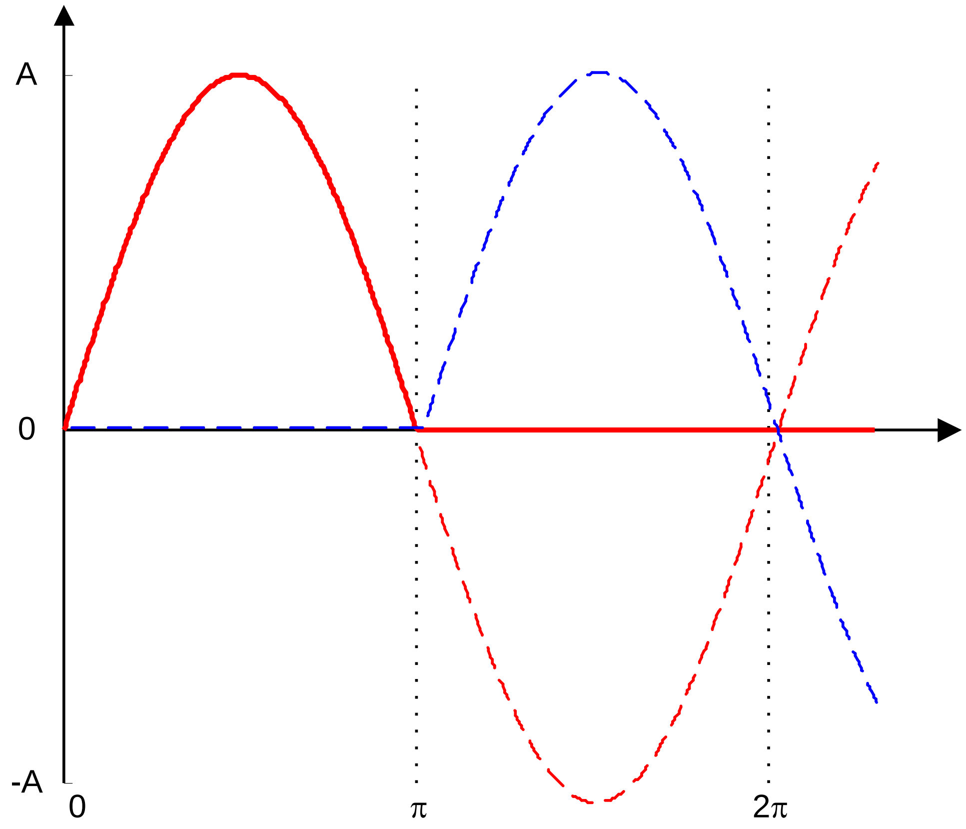


$$f(t)=h(t)-2h(t-a)+h(t-2a)$$

$$F(s)=L\{f(t)\}=\frac{1}{s}\left(1+e^{-2as}-2e^{-as}\right)$$

Primer – složen signal 2

$$h(t) = \begin{cases} 0, & t < 0 \\ A\sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

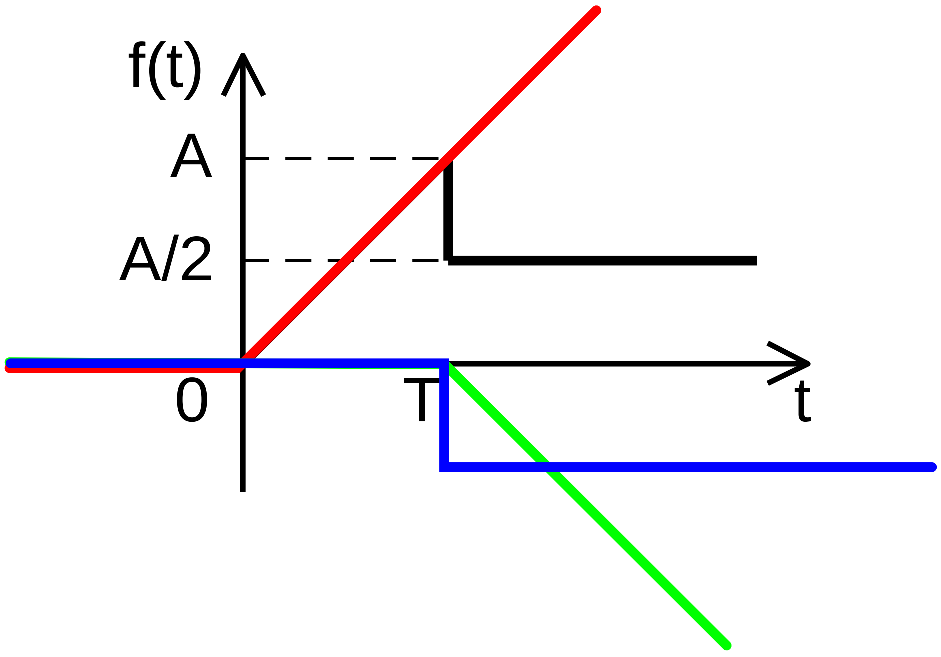


$$f(t) = A\sin t \cdot h(t) - A\sin(t-\pi)\cdot h(t-\pi)$$

$$F(s) = A\left(\frac{1}{s^2+1} + \frac{e^{-s\pi}}{s^2+1}\right) = \frac{A}{s^2+1}\left(1+e^{-s\pi}\right)$$

Primer – složen signal 3

$$h(t)=\begin{cases}0, & t<0\\ \dfrac{A}{T}t, & 0\le t<T\\ \dfrac{A}{2}, & t\ge T\end{cases}$$



$$f(t)=\frac{A}{T}t\cdot h(t)-\frac{A}{2}h(t-T)-\frac{A}{T}(t-T)\cdot h(t-T)$$

$$F(s)=\frac{A}{Ts^2}\left(1+e^{-sT}\right)-\frac{A}{2s}e^{-sT}$$

Inverzna Laplasova transformacija

$$F(s) = \frac{P(s)}{Q(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

- Za određivanje inverzne Laplasove transformacije su od posebnog značaja polovi funkcije *F*(*s*), i tu se mogu uočiti četiri karakteristična slučaja:
  - Svi polovi funkcije *F*(*s*) su realni i prosti
  - Funkcija *F*(*s*) ima višestruke realne korene
  - Postoje konjugovano-kompleksni polovi, a realni su, ako postoje, prosti
  - Funkcija *F*(*s*) ima višestruke konjugovano kompleksne polove

Polovi funkcije su realni i prosti

$$F(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{(s-s_1)(s-s_2)\ldots(s-s_n)}$$

$$F(s) = \frac{K_1}{s-s_1} + \frac{K_2}{s-s_2} + \ldots + \frac{K_n}{s-s_n} = \sum_{k=1}^{n} \frac{K_k}{s-s_k}$$

$$K_k = \left[(s-s_k)\frac{P(s)}{Q(s)} \right]_{s=s_k}$$

$$K_k = \lim_{s \to s_k} \left[(s-s_k)\frac{P(s)}{Q(s)} \right] = \lim_{s \to s_k} \frac{\frac{d}{ds}(s-s_k)P(s)}{\frac{d}{ds}Q(s)} = \frac{P(s_k)}{Q'(s_k)}$$

$$f(t) = L^{-1}\left[\sum_{k=1}^{n} \frac{K_k}{s-s_k} \right] = \sum_{k=1}^{n} K_k \cdot e^{s_k t}, \quad t > 0$$

Polovi funkcije su realni višestruki

$$F(s)=\frac{P(s)}{Q(s)}=\frac{P(s)}{(s-s_1)^3(s-s_4)\dots(s-s_n)}$$

$$F(s)=\frac{K_{11}}{(s-s_1)^3}+\frac{K_{12}}{(s-s_1)^2}+\frac{K_{13}}{(s-s_1)}+\sum_{k=4}^{n}\frac{K_k}{s-s_k}$$

$$(s-s_1)^3\frac{P(s)}{Q(s)}=K_{11}+(s-s_1)K_{12}+(s-s_1)^2K_{13}+(s-s_1)^3\sum_{k=4}^{n}\frac{K_k}{s-s_k}$$

$$K_{11}=\left[(s-s_1)^3\frac{P(s)}{Q(s)}\right]_{s=s_1}$$

$$K_{12}=\left[\frac{d}{ds}(s-s_1)^3\frac{P(s)}{Q(s)}\right]_{s=s_1}$$

$$K_{13}=\frac{1}{2}\left[\frac{d^2}{ds^2}(s-s_1)^3\frac{P(s)}{Q(s)}\right]_{s=s_1}$$

$$K_{rm}=\frac{1}{(m-1)!}\left[\frac{d^{m-1}}{ds^{m-1}}(s-s_r)^p\frac{P(s)}{Q(s)}\right]_{s=s_r}$$

$$m=1,2,\dots p$$

$$f(t)=\frac{K_{11}}{2}t^2e^{s_1t}+K_{12}te^{s_1t}+K_{13}e^{s_1t}+\sum_{k=4}^{n}K_k\cdot e^{s_kt},\quad t>0$$

Polovi funkcije su konjugovano kompleksni

$$F(s)=\frac{P(s)}{Q(s)}=\frac{P(s)}{(s-s_1)(s-s_1^*)(s-s_3)\ldots(s-s_n)}$$

$$F(s)=\frac{K_1}{s-s_1}+\frac{K_1^*}{s-s_1^*}+\frac{K_3}{s-s_3}+\ldots+\frac{K_n}{s-s_n}$$

$$s_1=-\alpha+j\omega,\quad s_1^*=-\alpha-j\omega$$

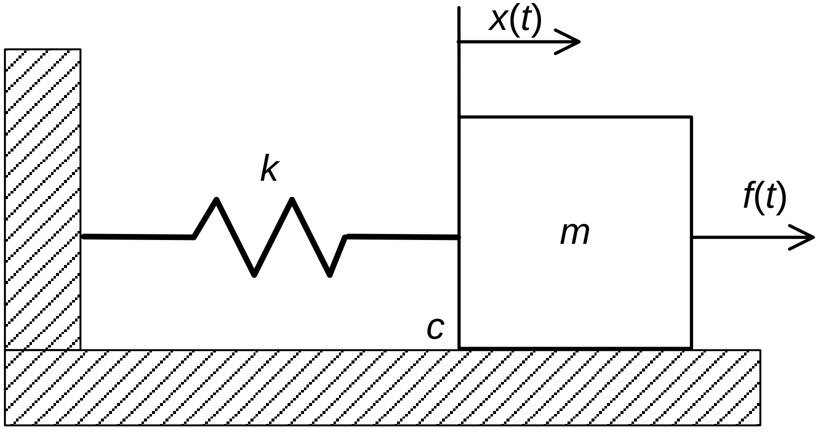
$$K_1=a+jb,\quad K_1^*=a-jb$$

$$K_1=a+jb=\left.\frac{P(s)}{Q'(s)}\right|_{s=-\alpha+j\omega}$$

$$F(s)=\frac{a+jb}{(s+\alpha)-j\omega}+\frac{a-jb}{(s+\alpha)+j\omega}+\sum_{k=3}^{n}\frac{K_k}{s-s_k}=\frac{2a(s+\alpha)-2b\omega}{(s+\alpha)^2+\omega^2}+\sum_{k=3}^{n}\frac{K_k}{s-s_k}$$

$$f(t)=2a\cdot e^{-\alpha t}\sin\omega t-2b\cdot e^{-\alpha t}\cos\omega t+\sum_{k=3}^{n}K_k e^{s_k t},\quad t>0$$

Primer primene Laplasove transformacije



- Mehanički sistem je opisan diferencijalnom jednačinom

$$m\frac{d^2x(t)}{dt^2}+c\frac{dx(t)}{dt}+k\cdot x(t)= f(t)$$

$$f(t)=0,\quad x(0_-)=x_0=1,\quad \frac{dx}{dt}(0_-)=0$$

Primer – nastavak

- Primena Laplasove transformacije na dif. jedn. daje:

$$m\left(s^2 X(s) - sx(0_-) - \frac{dx}{dt}(0_-)\right) + c\big(sX(s) - x(0_-)\big) + k \cdot X(s) = 0$$

- a nakon sređivanja:

$$\left(ms^2 + cs + k\right)X(s) = (ms + c)x(0_-)$$

$$X(s) = \frac{ms + c}{ms^2 + cs + k}x(0_-) = \frac{ms + c}{ms^2 + cs + k}$$

$$X(s) = \frac{P(s)}{Q(s)} = \frac{s + \frac{c}{m}}{s^2 + \frac{c}{m}s + \frac{k}{m}}$$

Primer – realni jednostruki polovi

- Za k/m=2 i c/m=3

Razvoj u sumu parcijalnih sabiraka

$$X(s)=\frac{s+3}{s^2+3s+2}=\frac{s+3}{(s+1)(s+2)}=\frac{2}{s+1}-\frac{1}{s+2}$$

- Original x(t) se dobija promenim inverzne Laplasove transformacije (upotrebom tablica)

$$x(t)=L^{-1}\{X(s)\}=L^{-1}\left\{\frac{2}{s+1}\right\}-L^{-1}\left\{\frac{1}{s+2}\right\}=2e^{-t}-e^{-2t}$$

```
K=2; M=1; c=3;
P=[M c]; Q=[M c K];
[nule,polovi,ostatak]=residue(P,Q)

plot(polovi+eps*j,'x')

roots(Q)
```

```
nule =
    -1
     2
polovi =
    -2
    -1
ostatak =
     []
```

Primer – realni višestruki polovi

- Za *k*/*m*=4 i *c*/*m*=4 Razvoj u sumu parcijalnih sabiraka

$$X(s) = \frac{s+4}{s^2+4s+4} = \frac{s+4}{(s+2)^2} = \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

- Original x(t) se dobija promenim inverzne Laplasove transformacije (upotrebom tablica)

$$x(t) = L^{-1}\{X(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} + L^{-1}\left\{\frac{2}{(s+2)^2}\right\} = e^{-2t} + 2te^{-2t}$$

```
K=4; M=1; c=4;
P=[M c]; Q=[M c K];
[nule,polovi,ostatak]=residue(P,Q)

plot(polovi+eps*j,'x')

roots(Q)
```

```
nule =
     1
     2
polovi =
    -2
    -2
ostatak =
     []
```

Primer – konjugovano-kompleksni polovi

- Za k/m=3 i c/m=2 Razvoj u sumu parcijalnih sabiraka

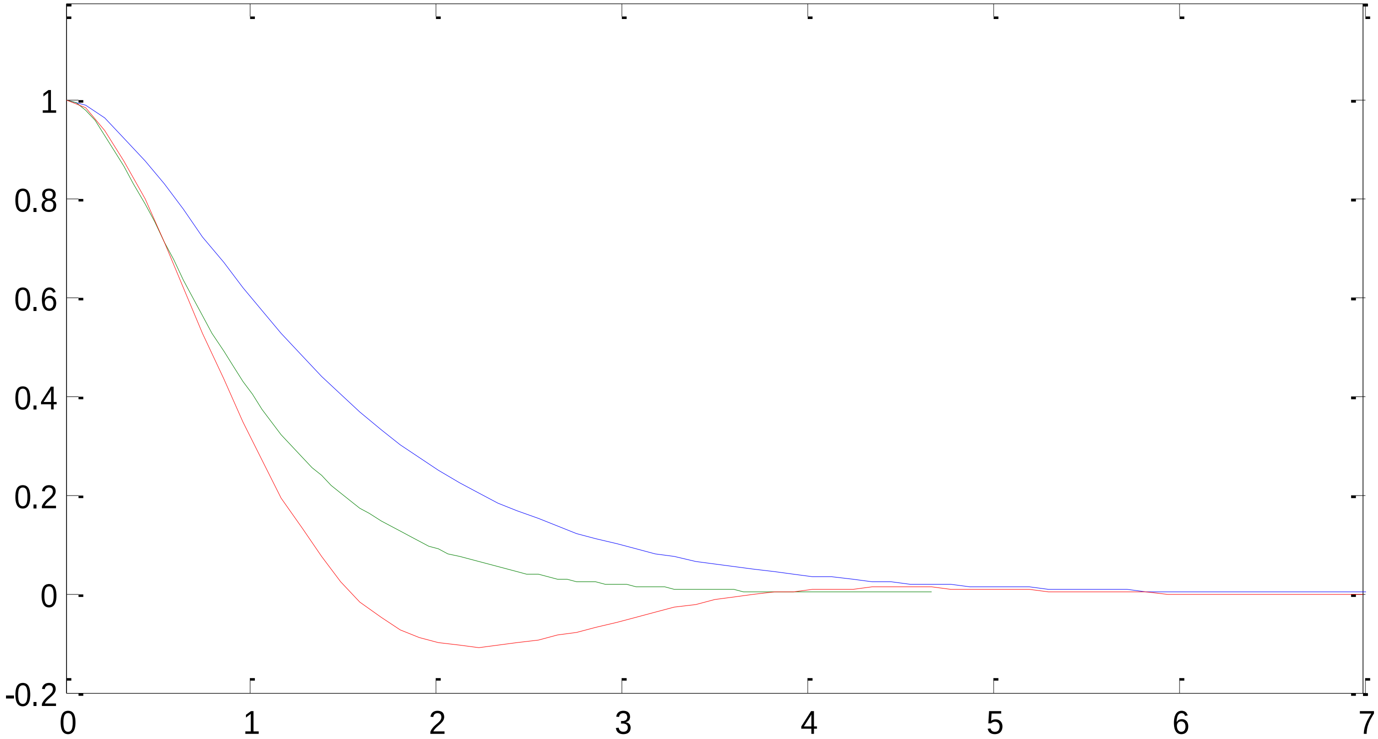
$$X(s)=\frac{s+2}{s^2+2s+3}=\frac{s+1+1}{(s+1)^2+(\sqrt{2})^2}=\frac{s+1}{(s+1)^2+(\sqrt{2})^2}+\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}$$

- Original x(t) se dobija promenim inverzne Laplasove transformacije (upotrebom tablica)

$$x(t)=L^{-1}\{X(s)\}=L^{-1}\left\{\frac{s+1}{(s+1)^2+(\sqrt{2})^2}\right\}+\frac{1}{\sqrt{2}}\cdot L^{-1}\left\{\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}\right\}$$

$$x(t)=e^{-t}\cos(t\sqrt{2})+\frac{1}{\sqrt{2}}e^{-t}\sin(t\sqrt{2})$$

Primer – uporedni prikaz



Odziv modela 2. reda

- Uvođenjem smena model se može napisati kao

$$\xi = \frac{c}{2\sqrt{km}}$$ ← Faktor relativnog prigušenja

$$\omega_n = \sqrt{\frac{k}{m}}$$ ← Prirodna učestanost

$$X(s) = \frac{ms + c}{ms^2 + cs + k} x_0 = \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2} x_0$$

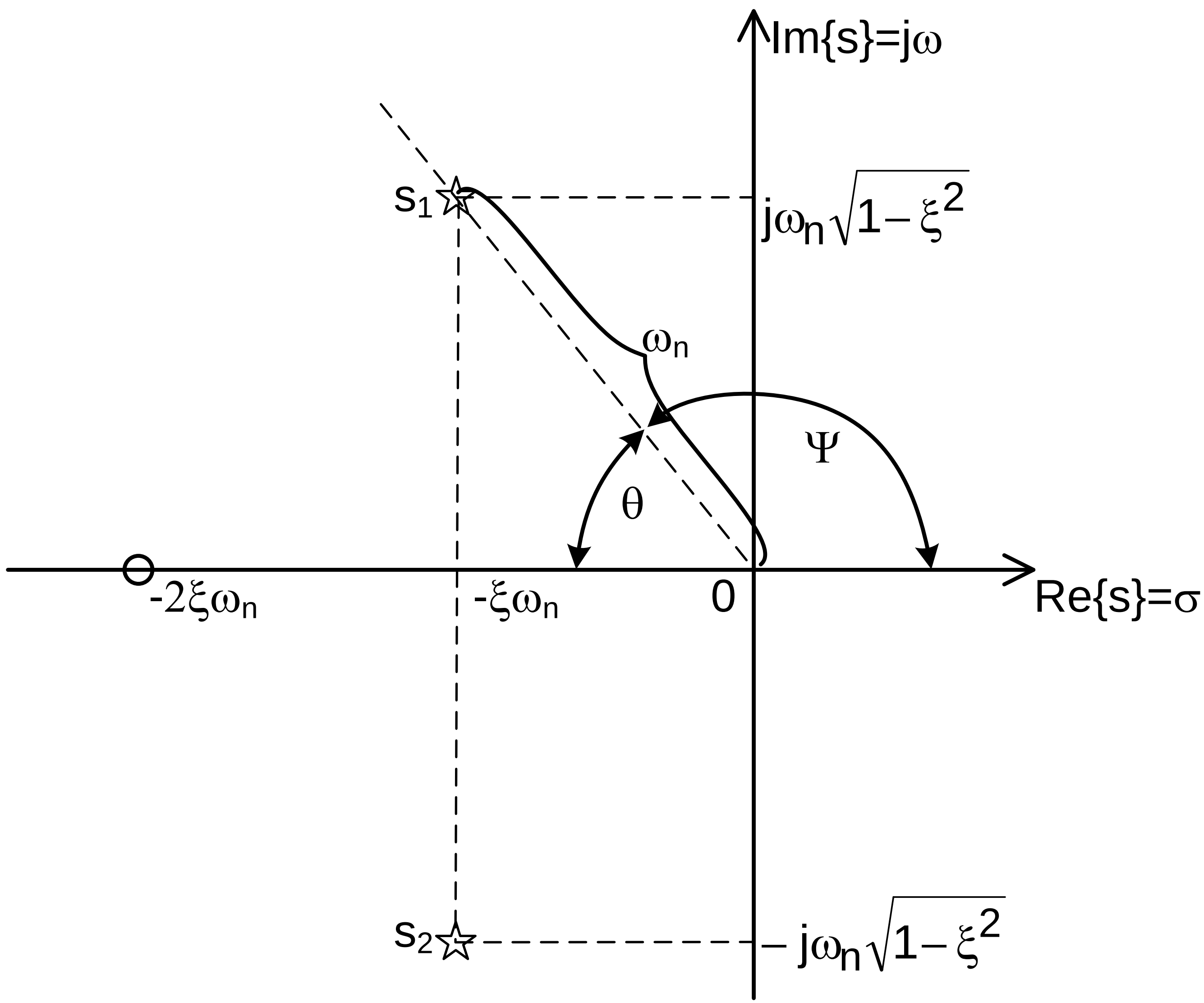
- Na karakter odziv sistema utiču polovi sistema

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

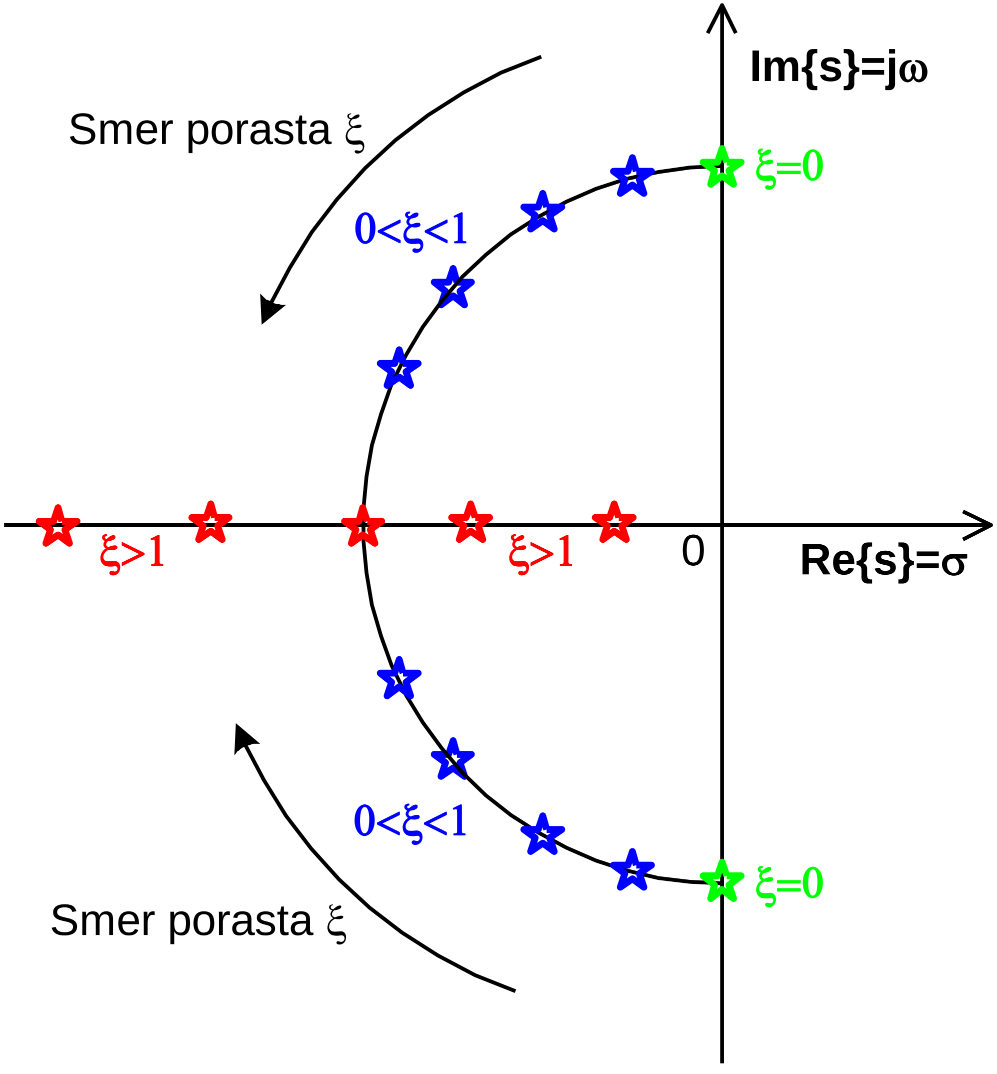
$$s_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$ ← Realni koreni $\xi \geq 1$

$$s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1 - \xi^2}$$ ← Konjugovano-kompleksni polovi $\xi < 1$

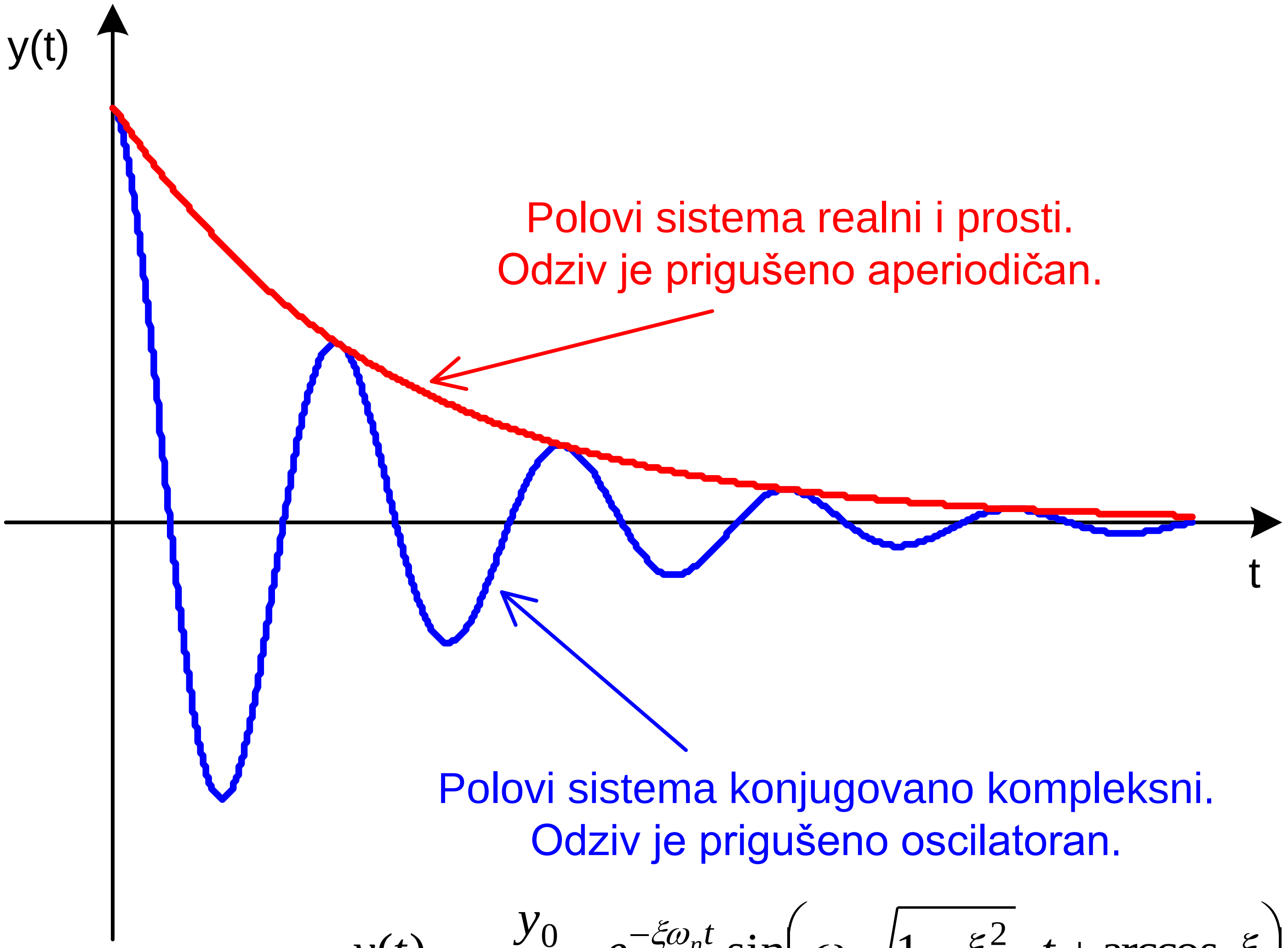
Lokacije konjugovano-kompleksnih polova



Uticaj ξ na lokacije polova



Prigušen odziv sistema



$$y(t) = \frac{y_0}{1-\xi^2} e^{-\xi\omega_n t} \sin\left(\omega_n \sqrt{1-\xi^2} \cdot t + \arccos \xi \right)$$

Osobine linearnih modela

- **Princip superpozicije**. Odziv linearnog sistema na pobudu datu zbirom pojedinačnih pobuda može se dobiti kao suma odziva na pojedinačne pobude, koje na sistem deluju nezavisno jedna od druge.

- **Princip stacionarnosti**. Ako na linearan, stacionaran sistem bez početne energije (nulti početni uslovi) deluje pobuda $x(t)h(t)$ i odziv na tu pobudu je $y(t)h(t)$, tada će za čisto vremenski zakašnjenu pobudu $x(t-T)h(t-T)$ sistem imati odziv $y(t-T)h(t-T)$.
  - Ova se osobina još naziva i nezavisnost početka računanja vremena.

Primer – pobuda h(t)

```
>> K=2; M=1; c=3;
>> P=[1]; Q=[M c K 0];
>> [nule,polovi,ostatak]=residue(P,Q)
nule =
    0.5000
   -1.0000
    0.5000
polovi =
    -2
    -1
     0
ostatak =
     []

>> t=0:0.01:10; y=0.5+0.5*exp(-2*t)-1*exp(-t);
>> plot(t,y)
>> step(1,[M c K])     % kasnije
```