Predmet:

Analiza linearnih sistemov

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Vrsta gradiva:

Zapiski predavanj

Avtor:

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Študijsko leto:

2015/16

ANALIZA SISTEMOV

Franjo Pernuš, Tomaž Vrtovec (4. nad.), lab. za slikovne tehnologije

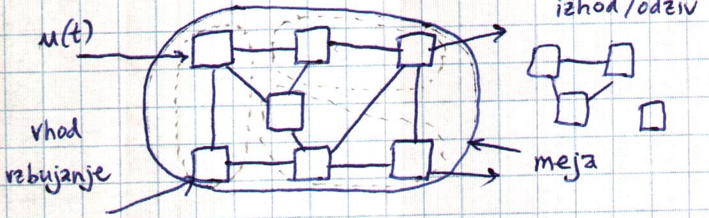
Senzum

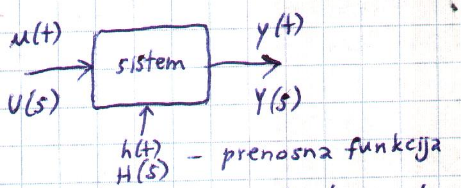
namest lab. vaj je seminar

izpit : 4 naloge (1 mora biti čist prav)
v enem tednu še ustni

I. UVOD

Definicija sistema : Sistem je skupina po naravnih zakonih povezanih in soodvisnih komponent (enot, elementov...), ki sestavljajo celoto.





Sistem je skupina med seboj načrtno povezanih soodvisnih komponent, ki sestavljajo zaključeno celoto.

Sistem je skupek z določenim namenom in po določenih načelih in lastnostih povezanih soodvisnih enot, ki sestavljajo zaključeno celoto.

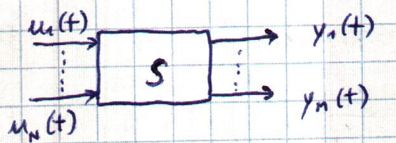
Sistem je množica povezanih komponent, ki tvorijo celoto, ki ima drugačne lastnosti kot posamezne komponente.

Sistem je celovitost urejene in omejene množice elementov, med katerimi obstajajo odnosi ali pa je odnose mogoče vzpostaviti.

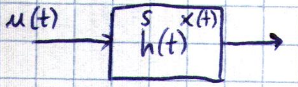
Meja loči sistem od okolice. Meja sistema omejuje oz. določa komponente znotraj in zunaj sistema. Meje sistema so lahko dejanske (realne) ali namišljene in so elastične (lahko se premika v obe smeri - oženje, širjenje).

S premikanjem mej se lahko osredotočimo samo na en del sistema, ki ga imenujemo **podsistem** in ga obravnavamo kot zaključen sistem.

V sistem lahko vključujemo tudi nove elemente in na ta način razširjamo sistem.



$u(t)$ $U(s)$
$y(t)$ $Y(s)$
$h(t)$ $H(s)$



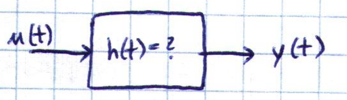
$x(t)$ ... notranja stanja sistema

① $y(t) = ?$ $u(t)$, $h(t)$ ... podana

$t \geqq 0$

$y(0)$, $y'(0)$...
analiza sistemov

② $h(t) = ?$
$H(s) = ?$ $u(t)$, $y(t)$ ... podana



~~sinteza sistemov~~

diferencialna / diferenčna enačba — zvezni s. / diskretni s.

→ zvezni s.

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \ldots + b_1 \frac{du}{dt} + b_0 u$$

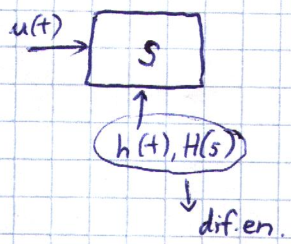
$y(t) = ?$ $t \geq 0$ $y(0), y'(0), \ldots y^{(n-1)}(0)$ zač. pogoji glede na red dif. enačbe

$u(t)$, $h(t)$ sistem spravimo v enačbo, mat. izraz

→ diskretni s. (s. opišemo z diferenčno enačbo)

$$a_n y[k+n] + a_{n-1} y[k+n-1] + \ldots + a_1 y[k+1] + a_0 y[k] =$$
$$= b_m u[k+m] + b_{m-1} u[k+m-1] + \ldots + b_1 u[k+1] + b_0 u[k]$$

zač. pogoji: $y[0], y[1], \ldots, y[n-1]$ $y[k] = ?$ $k \geq 0$



$b_0 u$ / $\mathcal{L}$

$$a_n s^n Y(s) + \ldots + a_1 s Y(s) + a_0 Y(s) = b_m s^m U(s) + \ldots + b_1 s U(s) + b_0 U(s)$$
$$(a_n s^n + \ldots + a_1 s + a_0) Y(s) = (b_m s^m + \ldots + b_1 s + b_0) U(s)$$
$$Y(s) = \underbrace{\frac{b_m s^m + \ldots + b_1 s + b_0}{a_n s^n + \ldots + a_1 s + a_0}}_{H(s)} \cdot U(s)$$

$$Y(s) = H(s) \cdot U(s)$$

$$H(s) = \frac{Y(s)}{U(s)}$$

$$h(t) = \mathcal{L}^{-1}[H(s)]$$

I.2. LASTNOSTI SISTEMOV

a) Naravni / Umetni

b) Fizični / Konceptualni

c) Statični / Dinamični

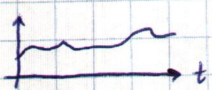
d) Odprti / Zaprti
↳ interakcija z okoljem je velika

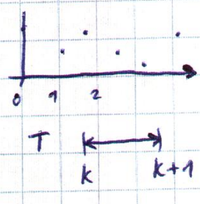
e) Enostavni / Kompleksni
↳ sistem sestavlja manjše število komponent

---

a) Stohastični / Deterministični
↳ parametri / spremenljivke naključno varirajo, se spreminjajo, tudi vhodni signali so lahko naključni (veter, ki vpliva na let)
↳ parametri in vh. signali so določljive veličine

b) Zvezni / Diskretni
↳ zvezna časovna skala





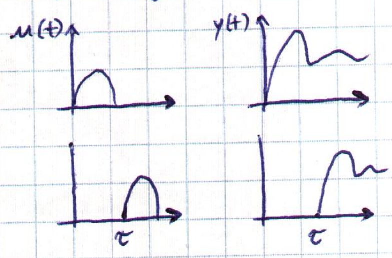
c) Linearni / Nelinearni $\quad u(t) \to \boxed{} \to y(t)$

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = u(t)$$
↑
R, L, C

$u_1(t) \to y_1(t)$
$u_2(t) \to y_2(t)$
$(a_1 u_1(t) + a_2 u_2(t)) \to (a_1 y_1(t) + a_2 y_2(t))$ linearen
$\neq$ nelinearen

d) Časovno nespremenljivi / Časovno spremenljivi

$u(t) \to y(t)$
$u(t-\tau) \to y(t-\tau)$



~~linearen~~

~~nelinearen~~

razberemo iz koeficientov, če so konstante je nespremenljiv, če so funkcije časa je sistem spremenljiv

e) Homogeni / Nehomogeni
↳ sistemi z vzbujanjem

f) Strnjeni / Porazdeljeni

lahko uporab. Kirch...

$\lambda = \frac{v}{f}$ $\quad v = 10^8$ m/s
$f = 10^6$

$\lambda = \frac{10^8}{10^6} = 100$ m

Stabilni / Nestabilni

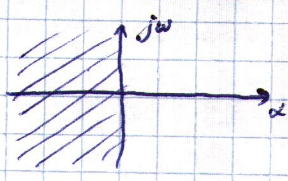
$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = u(t) \quad / \mathcal{L}$$

$$a_2 s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = U(s)$$
$$(a_2 s^2 + a_1 s + a_0)\, Y(s) = U(s)$$

$$\frac{Y(s)}{U(s)} = H(s) = \frac{1}{a_2 s^2 + a_1 s + a_0} = \frac{P(s)}{Q(s)}$$

$Q(s)$ ... značilni / karakteristični polinom
$Q(s) = 0$ značilna enačba
↳ s_1, s_2

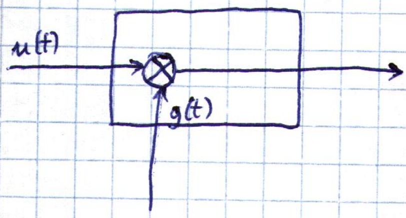
če so vsi koreni na levi → stabilen



$y(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$Re[S_i] \leq 0$

Primer: Je linearen sistem? ✓



$y(t) = u(t) \cdot g(t)$

splošno za lin: $u_1 \to y_1$

$u_2 \to y_2$

$(a_1 u_1 + a_2 u_2) \to (a_1 y_1 + a_2 y_2)$

$u_1 \to u_1 \cdot g$

$u_2 \to u_2 \cdot g$

pričak. $u_1 + u_2 \to u_1 g + u_2 g = (u_1 + u_2) \cdot g$

dejansko $(u_1 + u_2) g = y(t)$

Je časovno spr.? ✓

$u(t) \to y(t)$

$u(t-\tau) \to y(t-\tau)$

Primer: $y(t) = u^2(t)$ DN

Lin
čas. spr.

Primer: Je lin.?

$\underline{y(t) = u(t) + K}$

$u_1 \quad y_1 = u_1 + K$

$u_2 \quad y_2 = u_2 + K$

$(u_1 + u_2) + K$

$\neq$

prič. $y = y_1 + y_2 = (u_1 + K) + (u_2 + K)$

① $\frac{dy}{dt} = 5y(t) \qquad \frac{dy}{dt} - 5y(t) = 0$ — h., l., č.n., strnjen, 1. reda

② $\frac{dy}{dt} = 5y^2(t)$ — h., nelin., č.n., str.

③ $\frac{dy}{dt} = 5 \cdot y(t) + \underline{4t}$ — neh., l., č.n., str.

④ $\frac{dy}{dt} = (9t+1) y^2(t) + 5t$ — neh., nelin., č.s., str.

⑤ $\frac{dy}{dt} = e^{y(t)} + 5y(t) + \overset{u}{g}(t)$ — neh., nelin, č.n., str.

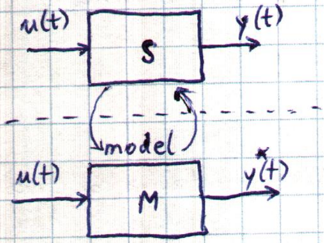
⑥ $\frac{\partial}{\partial t} y(z,t) = \frac{\partial^2}{\partial z^2} y(z,t) + z\,y(z,t)$ — h., l., č.n., porazd.

↳ 2 neznanki

⑦ $\frac{dy}{dt} + \frac{dx}{dt} = 8x(t) - 6y(t)$

$\frac{dy_1}{dt} + \frac{dy_2}{dt} = 8y_2(t) - 6y_1(t)$ — h., l., č.n., str. (sistem ima dva izhoda)

II. MODELIRANJE SISTEMOV



$y^* \approx y$

Def.:

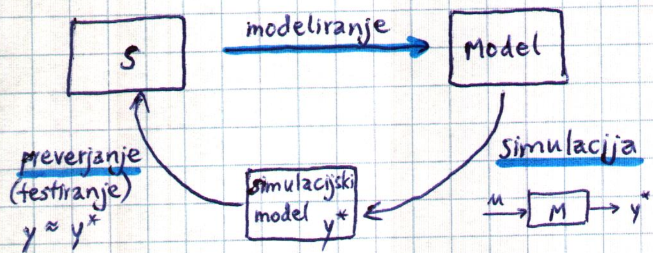
Model je pomanjšana kopija nečesa (sistema).

Model je poseben tip oz. verzija nekega izdelka.

Model je množica idej in številk, ki opisujejo pretekla, sedanja in prihodnja stanja sistema ali procesa, ki pomagajo pri izračunih in napovedih.

Model je aproksimacija ali simulacija realnega sistema, ki izpušča vse, razen najpomembnejših spremenljivk sistema.

Model je predstavitev sistema, ki omogoča preučevanje sistema ter napovedovanje obnašanja sistema v prihodnosti.



Matematični model

$$a_2\ddot{y} + a_1\dot{y} + a_0 y = u(t)$$

$y = ?$

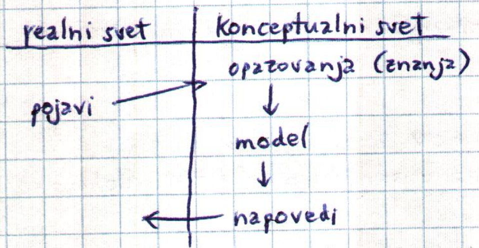
$y(t) = n(t)$

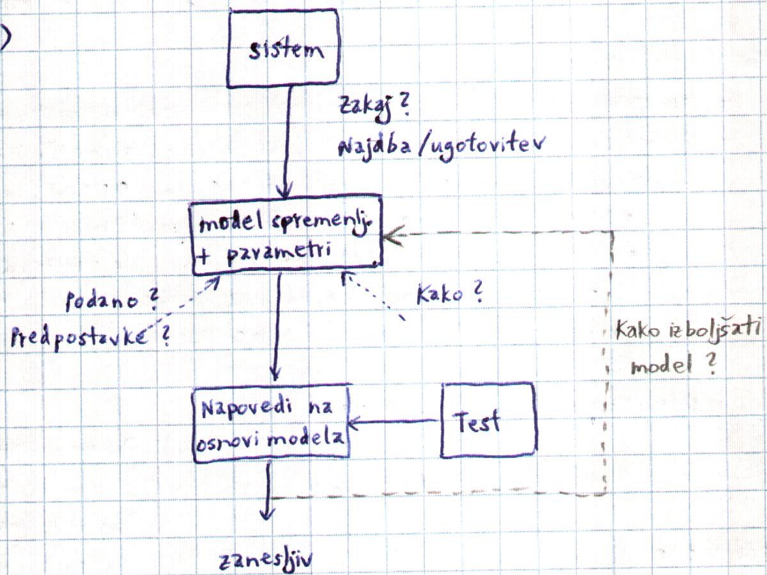
$$\frac{dy(t)}{dt} = \frac{dn(t)}{dt} = \gamma \cdot n(t)$$

$y(0)$

$$\frac{dn(t)}{dt} - \gamma n(t) = 0$$

$n(t),\ t \geq 0$





Karba: Modeliranje procesov

Kunc: Zdravstveni vestnik 77, 2008

Modeliranje / Simulacija

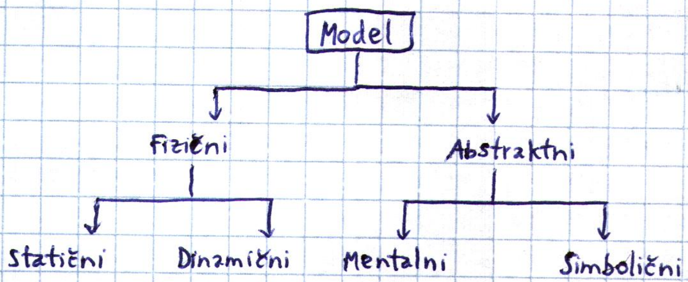
Modeliranje in simulacija sta dva tesno povezana postopka, ki obravnavata kompleksne aktivnosti modeliranja in eksperimentiranja z modeli.

Namen modeliranja

- napovedovati obnašanje sistema v različnih situacijah (vsaka napoved lahko predstavlja koristno info.)
- omogočiti načrtovanje sistemov vodenja in njihovo vrednotenje
- z modelom ocenimo parametre sistema, ki niso direktno merljivi
- lahko testiramo občutljivost sistemskih parametrov
- želimo optimirati obnašanje sistema
- omogočiti učinkovito odkrivanje napak v sistemu
- omogočiti raziskavo sistemov, ki bi bili v realnem svetu lahko dragi ali tvegani

Modeliranje je problemsko orientirano, simulacija pa je relativno neodvisna od področja.

Razvrstitev modelov



Fizični modeli so poenostavljene fizične predstavitve realnih sistemov. Njihova gradnja je velikokrat zahtevna, zamudna, draga in nepraktična.
- Statični so pomanjšane kopije realnih objektov (model avtomobila, letala, maketa hiše) ali imitacije.
- Dinamični so lahko analogni modeli ali prototipi pomanjšane kopije realnih sistemov.

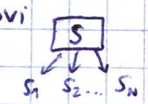
Abstraktni:
- Mentalni modeli so subjektivna slika realnih sistemov.
- Simbolični so veliko bolj uporabni in enostavnejši za gradnjo kot fizični. Delimo jih na nematematične, matematične in logične.
  - Nematem. delimo na verbalne, grafične in shematične.
  - Matem. so največkrat uporabljeni. Informacije, ki jih dobimo na osnovi mat. modela, je splošno razumljiva in koristna. Manipulacija in vrednotenje možnih različic je relativno poceni.

Načini modeliranja (matematičnega)

$$a_n \frac{d^n y}{dt^n} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_0 u$$

a) - teoretično
b) - eksperimentalno
c) - hibridno / kombinirano

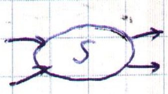
a) Pri teoretičnem modeliranju razčlenimo/razstavimo sistem/proces na večje število enostavnejših podsistemov. Povezave med podsistemi določimo na osnovi zakonov oz. znanja, ki veljajo na nekem področju. Povezave so v tehničnih znanostih dobro določene, na drugih področjih pa lahko določanje povezav predstavlja rešen problem. Precizen mat. model nekega sistema je lahko zelo kompleksen. Kompleksnost pa ponavadi zmanjšuje uporabnost modela.



Možna rešitev je premišljena poenostavitev, ki ne sme seči predaleč. Paziti je treba, da je poenostavljen model še vedno dobro opisuje realni sistem.

- linearnost
- časovna spremenljivost
- strnjenost
- red sistema (modela)
- determinističnost

Poiskati moramo kompromis med natančnostjo in kompleksnostjo modela.
Pomembna lastnost teoretičnih modelov je, da jih z manjšimi spremembami lahko uporabimo za modeliranje sorodnih problemov.



b) eksperimentalno

Pri tem načinu modeliranja gradimo mat. model na osnovi poskusov in meritev na realnem sistemu. Izbrati moramo vhode in izhode. Na vhodih spreminjamo signale, na izhodih pa merimo, kako se sistem odziva na vzbujanje. Na ta način iščemo mat. model, katerega rešitev bi dala rezultate, ki so čimbolj podobni izmerjenim veličinam na realnem sistemu.

c) kombinirano

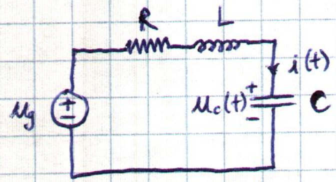
Združuje dobre lastnosti teoretičnega in eksp. modeliranja. Strukturo modela ponavadi določimo s teoretičnim modeliranjem, vrednosti parametrov pa z eksp. modeliranjem.
↳ $a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = u(t)$ ↳ a_0, a_1, a_2
Modeliranje je iterativen in interaktiven postopek.

Primeri modelov

$i_c = C \frac{du_c}{dt}$
$u_c = \frac{1}{C} \int$

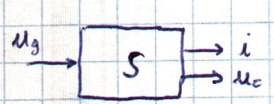
① Elektr.

KVZ: $u_g = u_R + u_L + u_C$
vejne enačbe: $u_R \leftrightarrow i_R$, $u_L \leftrightarrow i_L$, $u_C \leftrightarrow i_C$



$i = i_R = i_L = i_C$
$u_g = u_R + u_L + u_C$

$u_g = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \quad / \frac{d}{dt}$



$$\boxed{L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{du_g}{dt}}$$

$u_c = ?$
$u_g = u_c + L\frac{di}{dt} + Ri$
$i = C\frac{du_c}{dt}$

$$\boxed{u_g = u_c + LC\frac{d^2u_c}{dt^2} + RC\frac{du_c}{dt}}$$

$i = i_L = C\frac{du_c}{dt}$
$u_g = R\cdot i + L\frac{di}{dt} + u_c$

$L\frac{di}{dt} = -R\cdot i - u_c + u_g$
$C\frac{du_c}{dt} = i$

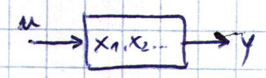
$$\boxed{\begin{aligned} \frac{di}{dt} &= -\frac{R}{L}i - \frac{1}{L}u_c + \frac{1}{L}u_g \\ \frac{du_c}{dt} &= \frac{1}{C}\cdot i \end{aligned}}$$

$$\begin{bmatrix} \frac{di}{dt} \\ \frac{du_c}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \cdot \begin{bmatrix} i \\ u_c \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u_g$$

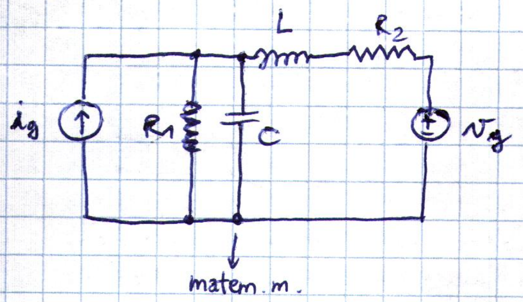
$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = u(t) \qquad \text{z eno DE}$$

$$\frac{dx_1}{dt} = \ldots \qquad \frac{dx_2}{dt} = \ldots \qquad \text{ali dvema}$$

$\longrightarrow x_1, x_2 \neq y$



Zgled:



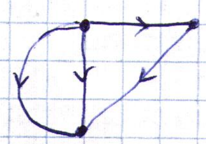
$u(t) \to i_g, v_g$

$y(t) \ldots$

$$x(t) = \begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix}$$

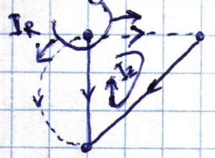
$$\frac{di_L}{dt} = f_1(i_L, v_c, v_g, i_g)$$

$$\frac{dv_c}{dt} = f_2(i_L, v_c, v_g, i_g)$$



usmerjen graf

Zvezje: sestavljajo veje, ki povezuje vsa vozlišča



zvezje •——• C, R

kite •------• L, R

zanka: L •------•

vse ostale •——•

prerez C •——•

vse ostale •------•

smer zanke = smer i_L

smer reza = smer i_C

$$I_Z: \quad v_L + v_{R_2} + v_g - v_c = 0$$

$$L \frac{di}{dt} + R_2 i_L + v_g - v_c = 0$$

$$① \quad \frac{di_L}{dt} = -\frac{R_2}{L} \cdot i_L + \frac{1}{L} v_c - \frac{1}{L} v_g$$

$$I_R: \quad i_c + i_L + i_{R_1} - i_g = 0$$

$$C \frac{dv_c}{dt} + i_L + \frac{v_c}{R_1} - i_g = 0$$

$$② \quad \frac{dv_c}{dt} = -\frac{1}{C} \cdot i_L - \frac{1}{R_1 C} \cdot v_c + \frac{1}{C} \cdot i_g$$

$$\underbrace{\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_c}{dt} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -\frac{R_2}{L} & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{R_1 C} \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} i_L \\ v_c \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}}_{B} \cdot \underbrace{\begin{bmatrix} i_g \\ v_g \end{bmatrix}}_{u}$$

$$\dot{x} = Ax + Bu \quad \Rightarrow x(t),\ t \geq 0$$

Dodatek:

$$y = \begin{bmatrix} i_{R1} \\ v_{R1} \end{bmatrix}$$

$$i_{R1} = g_1(i_L, v_C, i_g, v_g)$$

$$v_{R2} = g_2(i_L, v_C, i_g, v_g)$$

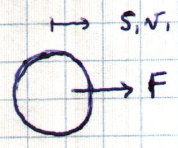
$$v_{R1} = v_C$$

$$i_{R1} = \frac{v_C}{R_1} \longrightarrow \begin{matrix} v_{R2} = R_2 \cdot i_{R2} \\ v_{R2} = R_2 \cdot i_L \end{matrix}$$

$$\underbrace{\begin{bmatrix} i_{R1} \\ v_{R2} \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} 0 & \frac{1}{R_1} \\ R_2 & 0 \end{bmatrix}}_{C} \cdot \underbrace{\begin{bmatrix} i_L \\ v_C \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} \emptyset \end{bmatrix}}_{D} \cdot \underbrace{\begin{bmatrix} i_g \\ v_g \end{bmatrix}}_{u}$$

$$y = Cx + Du \Rightarrow y(t),\ t \geq 0$$

Mehanski sistemi



$$v(t) = \frac{d}{dt} s(t)$$

$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

$$F = m \cdot a$$

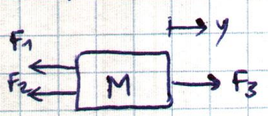
① translatorni
② rotacijski
③ ①+②

$y(t), x(t)$
$\downarrow$
$s(t), v(t), a(t)$
$\Downarrow$?

$\boxed{M}$	C		[kg]
k (vzmet)	L	k - koef. vzmeti	$\left[\frac{N}{m}\right]$
b (dušilka)	R	b - koef. dušenja	$\left[\frac{Ns}{m}\right]$

F... vzbujanje

$\boxed{M}$ diagram prostih teles

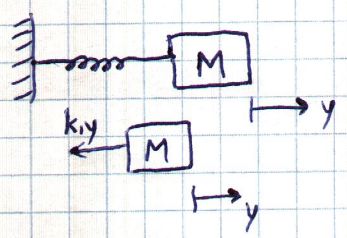


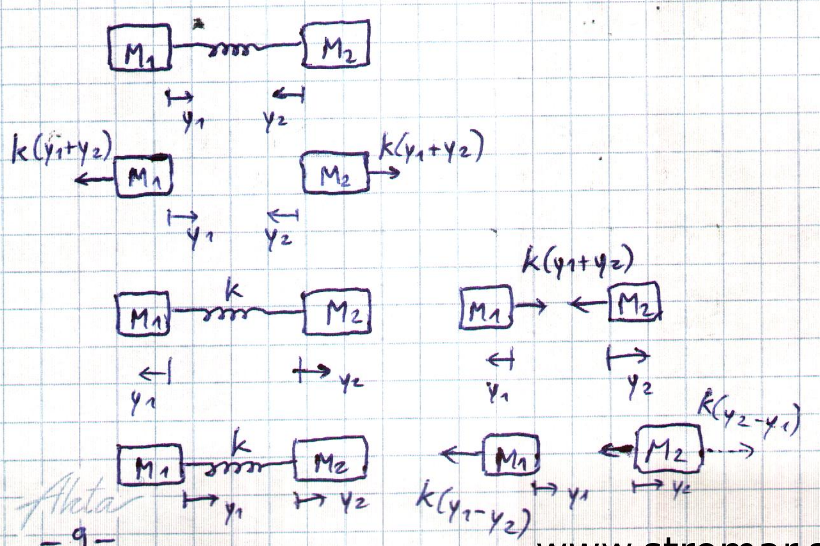
$$-F_1 - F_2 + F_3 = M \cdot a$$

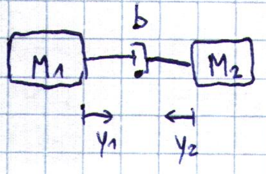
$$F_v = k \cdot \underset{\substack{x \\ y}}{s}$$

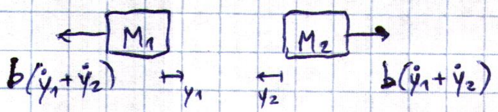
$$F_D = b \cdot v = b \cdot \frac{d\underset{\substack{x \\ y}}{s}}{dt}$$

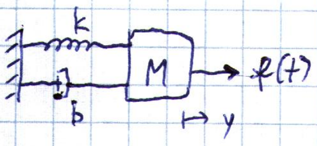
F_D: dušenje, trenje

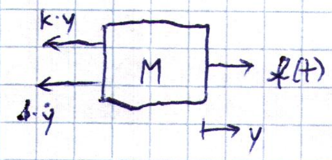










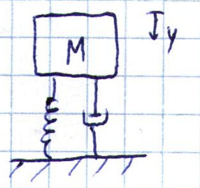


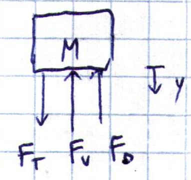
$$-k\cdot y - b\dot{y} + f(t) = M\cdot\ddot{y}$$

$$+M\ddot{y} + b\cdot\dot{y} - k\cdot y = f(t)$$

$$a_2\frac{dy^2}{dt^2} + a_1\frac{dy}{dt} + a_0 y = u(t)$$

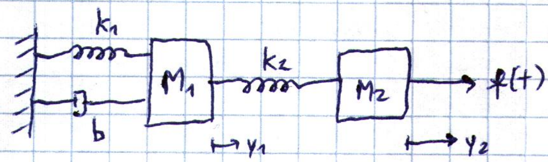
$$y(t) = ? \quad t \geq 0$$
$$y(0)$$

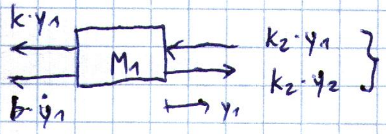


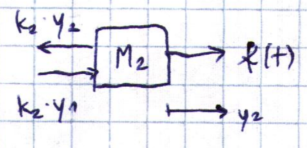


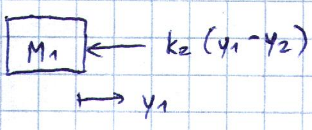
$$F_T - F_V - F_D = M\cdot a = M\cdot\ddot{y}$$

$$M\cdot g - k\cdot y - b\cdot\dot{y} = M\cdot\ddot{y}$$









$$k_2\cdot y_1 - k_2 y_2 + f(t) = M_2\ddot{y}_2$$

$$M_2\cdot\ddot{y}_2 + k_2 y_2 - k_2 y_1 = f(t)$$

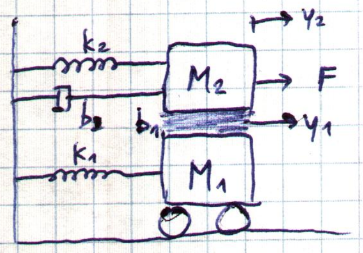
$$-k_1 y_1 - b\dot{y}_1 - k_2(y_1 - y_2) = M_1\cdot\ddot{y}_1$$

$$M_1\ddot{y}_1 + b\dot{y}_1 + (k_1+k_2)\cdot y_1 - k_2 y_2 = 0$$

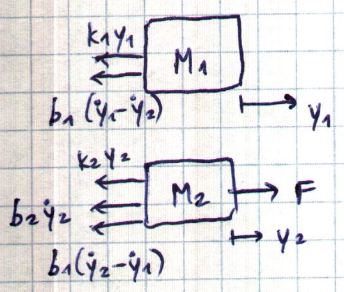
$$\frac{dy_1}{dt} = v_1$$
$$\frac{dy_2}{dt} = v_2$$

$$M_1\cdot\dot{v}_1 + b_1\cdot v_1 + (k_1+k_2)y_1 - k_2\cdot y_2 = 0$$
$$M_2\cdot\dot{v}_2 + k_2\cdot y_2 - k_2 y_1 = f(t)$$

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{v}_1\\ \dot{v}_2\end{bmatrix} = \begin{bmatrix} 4\times 4 \end{bmatrix}\cdot\begin{bmatrix}y_1\\ y_2\\ v_1\\ v_2\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\cdot f(t)$$



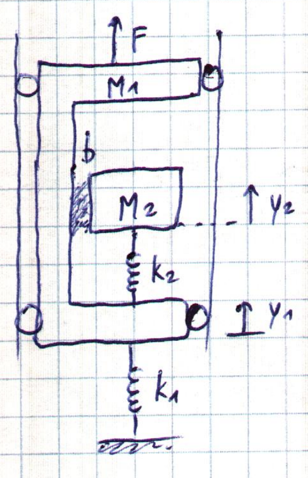
b_1 - koef. trenja



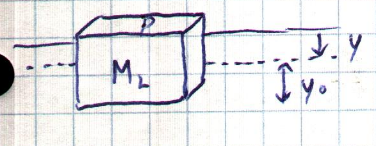
$$-b_1(\dot{y}_1-\dot{y}_2)-k_1\cdot y_1 = M_1\ddot{y}_1$$

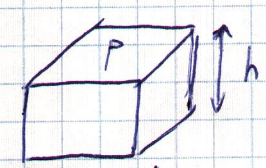
$$-k_2\cdot y_2-b_2\dot{y}_2-b_1(\dot{y}_2-\dot{y}_1)+F = M_2\ddot{y}_2$$

DN

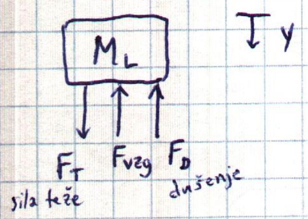


kocka ledu





$V = P\cdot h$



$$F_T - F_V - F_D = M_L\cdot\ddot{y}$$

$$M_L\cdot g - \rho_V\cdot P(y+y_0)g - b\cdot\dot{y} = \cancel{0}\, M_L\cdot\ddot{y}$$

$y(t) = ?$ $t \geq 0$

$y_0 = ?$

$$* \quad V = \frac{dy}{dt}$$

$$M_L\cdot\dot{V} + b\cdot V + \rho_V\cdot P\cdot g\cdot y = 0$$

$$F_T - F_V^* = 0$$

$$M_L\cdot\cancel{g} = \rho_V\cdot P\cdot y_0\cdot\cancel{g} = 0$$

$$y_0 = \frac{M_L}{\rho_V\cdot P} = \frac{\rho_L\cdot P\cdot h}{\rho_V\cdot P}$$

$$M_L\cdot g - \rho_V\cdot P\cdot y_0\cdot g - \rho_V\cdot P\cdot y\cdot g - b\dot{y} = M_L\ddot{y}$$

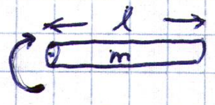
$$\boxed{M_L\ddot{y} + b\dot{y} + \rho_V\cdot P\cdot y\cdot g = 0}$$

ROTACIJSKI SISTEMI

Translacija		Rotacija	
Veličina	Enota	Veličina	Enota
masa (m, M)	kg	vztrajnostni moment J	$kg\,m^2$
sila (f, F)	N	navor τ	Nm
dolžina/pot (s, x, y)	m	kot θ	rad
hitrost $v = \dot{s}$	m/s	kotna hitr. $\omega = \dot{\theta}$	rad/s
pospešek $a = \dot{v} = \ddot{s}$	m/s^2	kotni posp. $\alpha = \dot{\omega} = \ddot{\theta}$	rad/s^2
koef. vzmeti k	N/m	koef. vzmeti k	Nm/rad
koef. dušenja b	Ns/m	koef. duš. b	Nms/rad

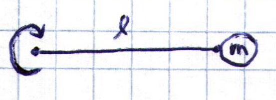
$J = \frac{1}{2} m r^2$

$J = \frac{2}{5} m r^2$



$J = \frac{1}{3} m l^2$

$J = \frac{1}{12} m l^2$



$J = m l^2$

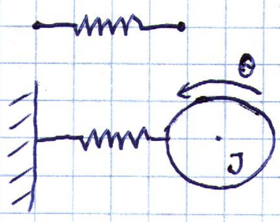
τ, J, α
$\tau = J \cdot \alpha$

f, m, a
$f = m \cdot a$

$\boxed{\Sigma \tau = J \cdot \alpha}$

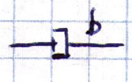
$\Sigma f = m \cdot a$

Vzmet



$\boxed{\tau = k \cdot \theta}$

Dušilka



$\boxed{\tau = b \cdot \dot{\theta} = b \cdot \omega}$

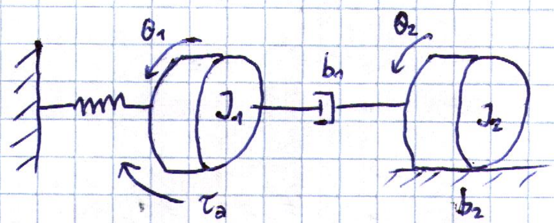
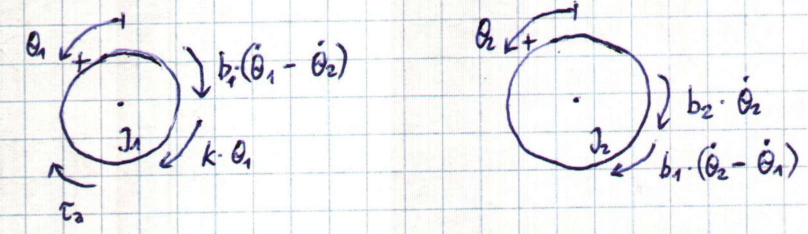


Diagram prostih teles



$$\Sigma\tau = J\cdot\alpha$$
$$\Sigma\tau_1 = J_1\cdot\alpha_1 = J_1\cdot\ddot\theta_1$$
$$\Sigma\tau_2 = J_2\cdot\alpha_2 = J_2\cdot\ddot\theta_2$$

① $-k\cdot\theta_1 - b_1(\dot\theta_1-\dot\theta_2)-\tau_a = J_{11}\ddot\theta_1$

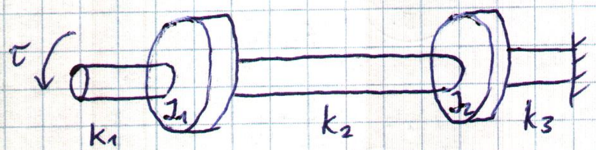
$(J_1\cdot\ddot\theta_1 + b_1\dot\theta_1 + k\cdot\theta_1) - b_1\dot\theta_2 = -\tau_a$

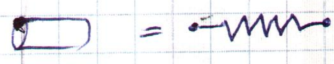
② $-b_1(\dot\theta_2-\dot\theta_1) - b_2\cdot\dot\theta_2 = J_2\cdot\ddot\theta_2$

$J_2\cdot\ddot\theta_2 + (b_1+b_2)\cdot\dot\theta_2 - b_1\dot\theta_1 = 0$

$\theta_1, \theta_2 = ?$

DN: Zmodeliraj !



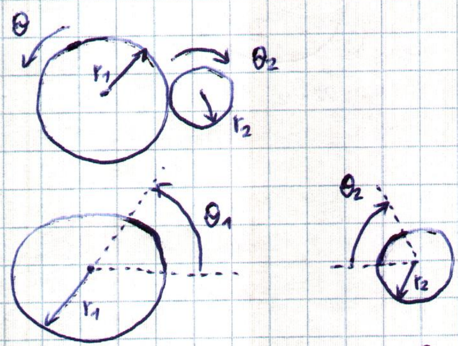


S samimi DE 1. reda

rešitev:

$$\frac{d}{dt}\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ -\frac{k_2}{J_1} & -\frac{b_1}{J_1} & \frac{k_2}{J_1} & 0\\ 0 & 0 & 0 & 1\\ \frac{k_2}{J_2} & 0 & -\frac{k_2+k_3}{J_2} & -\frac{b_2}{J_2}\end{bmatrix}\cdot\begin{bmatrix}\theta_1\\ \omega_1\\ \theta_2\\ \omega_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{J_1}\\ 0\\ 0\end{bmatrix}\tau$$

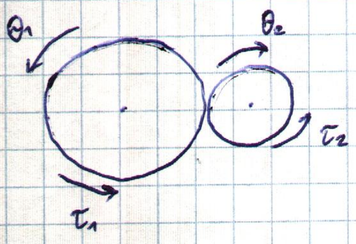
Zobniki

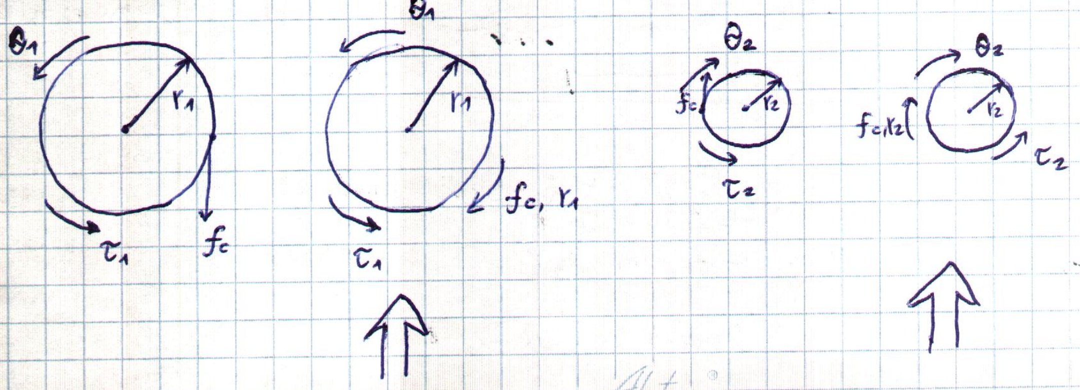


$r_1\cdot\theta_1$ $\qquad$ $r_2\cdot\theta_2$

$-r_2\theta_2$

$$\boxed{r_1\cdot\theta_1 = r_2\cdot\theta_2}$$





f_c - kontaktna sila

zobnik 1

$$\tau_1 - f_c \cdot r_1 = 0$$

$$\tau_1 = f_c \cdot r_1 \qquad f_c = \frac{\tau_1}{r_1}$$

zobnik 2

$$\tau_2 = f_c \cdot r_2 \qquad f_c = \frac{\tau_2}{r_2}$$

$$\boxed{\frac{\tau_1}{r_1} = \frac{\tau_2}{r_2}}$$

zgled:

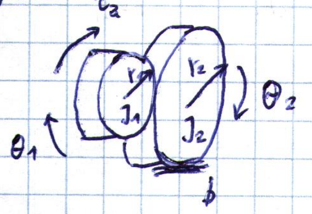
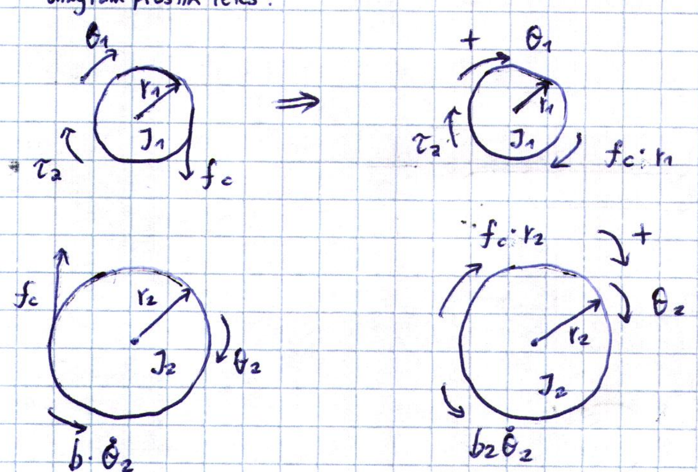


diagram prostih teles:



① $\tau_a + f_c \cdot r_1 = J_1 \alpha_1 = J_1 \ddot{\theta}_1$

② $-b \cdot \dot{\theta}_2 + f_c r_2 = J_2 \alpha_2 = J_2 \cdot \ddot{\theta}_2$

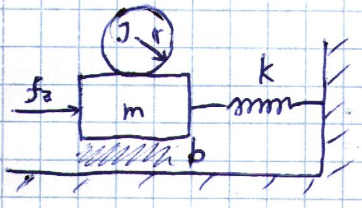
$$r_1 \theta_1 = -r_2 \theta_2$$

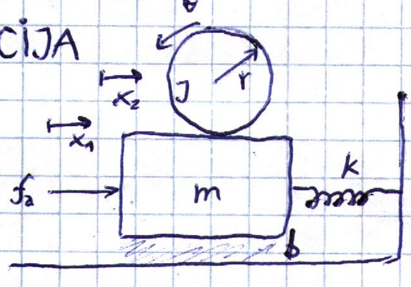
$$\boxed{\theta_2 = -\frac{r_1}{r_2} \cdot \theta_1}$$

② $f_c r_2 + b \cdot \frac{r_1}{r_2} \dot{\theta}_1 = -J_2 \cdot \frac{r_1}{r_2} \ddot{\theta}_1$

$$f_c = -\frac{J_2}{r_2} \frac{r_1}{r_2} \ddot{\theta}_1 - \frac{b}{r_2} \frac{r_1}{r_2} \dot{\theta}_1$$

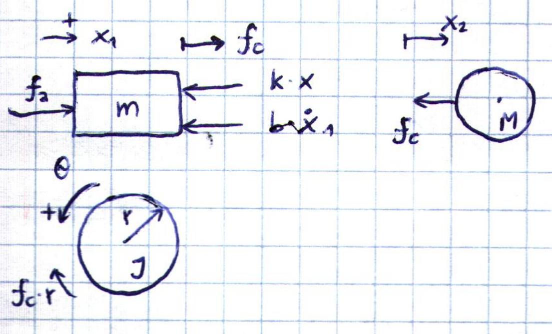
TRANSLACIJA + ROTACIJA





x_1, x_2, θ ?

rabimo 3 diagrame pr. teles

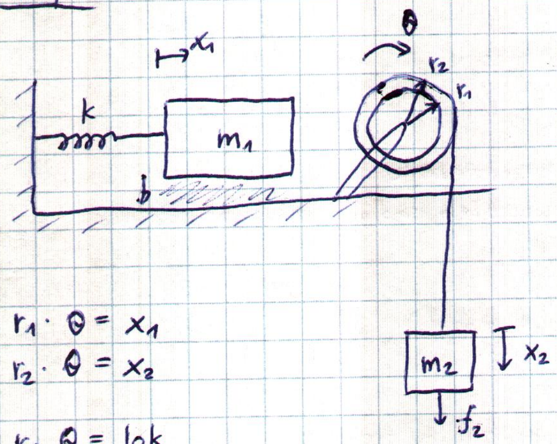


$$f_a + f_c - k \cdot x_1 - b \cdot \dot{x}_1 = m_1 \cdot \ddot{x}_1$$
$$-f_c \cdot r = J \cdot \ddot{\theta}$$
$$-f_c = m_2 \cdot \ddot{x}_2$$

neznanke: x_1, x_2, θ, f_c

$$\boxed{r \cdot \theta = x_1 - x_2}$$

Škripci

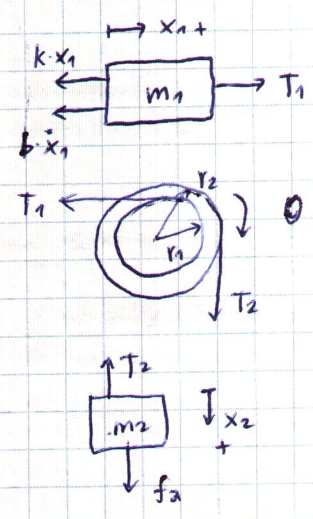


$r_1 \cdot \theta = x_1$

$r_2 \cdot \theta = x_2$

$r \cdot \theta = \text{lok}$

x_1, x_2, θ ?



DN 3 enačbe zapiš

Lotka, Voltera – sistem, ki opisuje sožitje živali v naravi (primer iz ekologije)

12.11. L (lisice), Z (zajci) – zanima nas, kako se spreminja njihovo število s časom

$L(t) = ?$ $\quad Z(t) = ?$

② $\dfrac{dL(t)}{dt} = c \cdot L(t) + d \cdot L(t) \cdot Z(t)$

ko je preveč L, ni več hrane → t je negativen ↓ odvod je neg.

① $\dfrac{dZ(t)}{dt} = a \cdot Z(t) + b \cdot L(t) \cdot Z(t)$

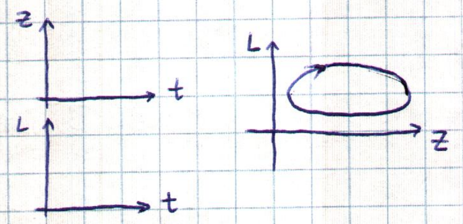
↑ če je neg., se št. Z manjša z naraščanjem št. L

$b < 0$ $\quad L(0)$

$c < 0$ $\quad Z(0)$

$\cancel{\begin{bmatrix} \frac{dZ}{dt} \\ \frac{dL}{dt} \end{bmatrix} = \begin{bmatrix} \; \end{bmatrix} \cdot \begin{bmatrix} Z \\ L \end{bmatrix} + \begin{bmatrix} \; \end{bmatrix} \cdot u(t)}$ ← ne moremo reševat z matrikami, ker je to samo za linearne sisteme

grafično:



niha število lisic in zajcev

ravnovesje (ni spremembe, odvod je 0), je v sistemih zelo pomembno

$a \cdot \bar{Z} + b \cdot \bar{L} \cdot \bar{Z} = 0$

$c \cdot \bar{L} + d \cdot \bar{L} \cdot \bar{Z} = 0$

$a, b, c, d = ?$

$\bar{Z}(a + b \bar{L}) = 0$

$\bar{L}(c + d \cdot \bar{Z}) = 0$

1.) $\bar{Z} = 0 \quad \bar{L} = 0$

2.) $\bar{Z} = 0 \quad (c + d\bar{Z}) = 0$

3.) $\bar{L} = 0 \quad (a + b\bar{L}) = 0$

4.) $(a + b\bar{L}) = 0 \quad (c + d\bar{Z}) = 0$

$\bar{L} = -\dfrac{a}{b}$

$\bar{Z} = -\dfrac{c}{d}$

pri linearnih sistemih imamo samo eno ravnovesno stanje, pri nelin. pa več.

eksperimentalno:
$a = 0{,}03$
$b = -0{,}001$
$c = -0{,}9$
$d = 0{,}0002$

$\bar{L} = 30$

$\bar{Z} = 450$

Richardsonov model oborožitve

$x(t)$ kako dobro je oborožena x skupina in kako y
$y(t)$

$$\boxed{\begin{aligned} \dot{x} &= a\cdot y - b\cdot x + g(t) \\ \dot{y} &= c\cdot x - d\cdot y + h(t) \end{aligned}}$$

+ oboroževanje, − razoroževanje
če je nasprotnik ful oborožen, smo mi vse bolj razoroženi

Samuelsonov inv. model (preprost investicijski model)

- govori o kapitalu in njegovem gibanju
- kapital, ki je na voljo nekemu gospodarstvu, podjetju, v trenutku

$K(t)$ ~ trenutni kapital (ki je na voljo)
K_e ~ ravnovesno stanje

$k(t) = K(t) - K_e$

$k(t) < 0$ (primankljaj)
≥ 0

$k(t),\ I(t)$ ← investicije

$$\frac{dk(t)}{dt} = I(t)$$

$$\frac{dI(t)}{dt} = -m\,k(t) - n\,I(t)$$

$$\frac{d^2k(t)}{dt^2} = -m\,k(t) - n\cdot\frac{dk}{dt}$$

$$\frac{d^2k}{dt^2} + n\frac{dk}{dt} + m\,k(t) = 0 \quad \Rightarrow k(t)$$

$k(0),\ k'(0) = I(0)$

Samuelsonov model narodnega dohodka (BDP)

BDP → $y[kT]$, $k = 0,1,2\ldots$
T - interval

$T = 3$ mesece

$y[k] = c[k] + i[k] + u[k]$

$c[k]$: izdatki potrošnikov (koliko potrošijo)
$i[k]$: kak industrije ustvarja
$u[k]$: država zapravlja

$u[k] = K$ (konstanta)

$c[k] = a\cdot y[k-1]$
$i[k] = b(c[k] - c[k-1])$
$i[k] = b(a\cdot y[k-1] - a\cdot y[k-2]) = a\cdot b\ (\sim)$

$y[k] = a\cdot y[k-1] + a\cdot b\,(y[k-1] - y[k-2]) + K$
$y[0], y[-1]$

↳ dif. e. drugega reda

① $\displaystyle \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0 y = b_0\cdot u(t)$

② $\displaystyle \frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_1 u$

$\displaystyle \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_2 u$

$\vdots$

$\displaystyle \frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_n u$

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \\ & & & \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \cdot u$$

$$\dot{x} = A \cdot x + B\cdot u$$

II.1. Prevedba diferencialne enačbe n-tega reda v sistemu n-diferencialnih enačb prvega reda

$x_1 = y$
$x_2 = \dot{y} = \dot{x}_1$
$x_3 = \ddot{y} = \ddot{x}_1 = \dot{x}_2$
$\vdots$
$x_n = y^{(n-1)}$

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \end{aligned}}$$

$x_n = ?$

$x_2 = \dot{y}$
$x_3 = \ddot{y}$
$x_{n-1} = y^{(n-2)}$
$x_n = y^{(n-1)}$
$\dot{x}_n = y^{(n)}$

$$\boxed{\dot{x}_n + a_{n-1} x_n + a_{n-2} x_{n-1} + \ldots + a_1 x_2 + a_0 x_1 = b_0 u}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & \vdots & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & \ldots & & -a_{n-1} \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} \cdot u$$

$\dddot{y} + 3\ddot{y} + 4\dot{y} + y = 7 \cdot u$

$\rightarrow x_1, x_2, x_3$

$$\begin{cases} x_1 = y \\ x_2 = \dot{y} = \dot{x}_1 \\ x_3 = \ddot{y} = \ddot{x}_1 = \dot{x}_2 \\ \dot{x}_3 = \dddot{y} \end{cases}$$

$\dot{x}_1 = x_2$
$\dot{x}_2 = x_3$
$\dot{x}_3 = -x_1 - 4x_2 - 3x_3 + 7 \cdot u$

$\dot{x}_3 + 3 \cdot x_3 + 4 \cdot x_2 + x_1 = 7 \cdot u$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -4 & -3 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} \cdot u$$

zač. pogoji: $x_1[0]$, $x_2[0]$, $x_3[0]$
$\dot{x} = Ax + Bu$
$x(t) = \ldots$

Sistem, ki ima več vhodov in izhodov

$\ddot{y}_1 + 3\dot{y}_1 = 4 \cdot u_1 - 2y_2$
$\ddot{y}_2 + y_2 = 5 \cdot u_2 - \dot{y}_1$

x_1, x_2, x_3, x_4 (ker imamo dve enačbi 2. reda)

$x_1 = y_1$
$\dot{x}_1 = \dot{y}_1 = x_2$
$\dot{x}_2 = \ddot{y}_1$

$\dot{x}_2 + 3x_2 = 4u_1 - 2x_3$

$x_3 = y_2$
$\dot{x}_3 = \dot{y}_2 = x_4$
$\dot{x}_4 = \ddot{y}_2$

$\dot{x}_4 + x_3 = 5u_2 - x_2$

$\dot{x}_1 = x_2$
$\dot{x}_2 = -3x_2 - 2x_3 + 4u_1$
$\dot{x}_3 = x_4$
$\dot{x}_4 = -x_2 - x_3 + 5u_2$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -3 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix} \cdot x + \begin{bmatrix} 0 & 0 \\ 4 & 0 \\ 0 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

drug sistem:

$$\ddot{y}_1 + 3\int y_1\,dt = 4\cdot u_1 - 2y_2$$
$$\ddot{y}_2 + y_2 = 5u_2 - \dot{y}_1$$
$$x_1, x_2, x_3, x_4, x_5 = ?$$

$\leftarrow$ 5 spremenljivk (integrala se znebimo tako, da enačbo še enkrat odvajamo)

① $\frac{d}{dt}$

$\dot{x} = Ax + Bu$ (kojo rešimo, dobimo x-e)

$y_1, y_2 = ?$

② $x_1 = \int y_1\,dt$

$\dot{x}_1 = y_1$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad y = Cx + Du$$

primer z vzbujanjem

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m u}{dt^m} + \dots + b_1\frac{du}{dt} + b_0 u$$

$m < n$

① predpostavimo, da na desni nimamo odvodov

$$\frac{d^n y}{dt^n} + \dots + a_1\frac{dy}{dt} + a_0 y = u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_0 & \dots & & & -a_{n-1} \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \cdot u \qquad \dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$x_1^{(n)} + a_{n-1}x_1^{(n-1)} + \dots + a_1\dot{x}_1 + a_0 x_1 = u$$

$\uparrow$ m-krat $\frac{d}{dt}$

① odvod $\quad \dot{x}_1^{(n)} + a_{n-1}\dot{x}_1^{(n-1)} + \dots + a_1\ddot{x}_1 + a_0\dot{x}_1 = \dot{u}$

$$x_2 = \dot{x}_1$$
$$x_2^{(n)} + a_{n-1}x_2^{(n-1)} + \dots + a_1\dot{x}_2 + a_0 x_2 = \dot{u}$$

② $\quad \dot{x}_2^{(n)} + a_{n-1}\dot{x}_2^{(n-1)} + \dots + a_1\ddot{x}_2 + a_0\dot{x}_2 = \ddot{u}$

$$x_3 = \dot{x}_2$$

ⓜ $\quad \dot{x}_m^{(n)} + a_{n-1}\dot{x}_m^{(n-1)} + \dots + a_1\ddot{x}_m + a_0\dot{x}_m = u^{(m)}$

$$x_{m+1} = \dot{x}_m$$

$$x_{m+1}^{(n)} + a_{n-1}x_{m+1}^{(n-1)} + \dots + a_1\dot{x}_{m+1} + a_0 x_{m+1} = u^{(m)}$$

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1\dot{y} + a_0 y = b_m\left[x_{m+1}^{(n)} + \dots + a_1\dot{x}_{m+1} + a_0 x_{m+1}\right] +$$
$$+ b_{m-1}[\quad] + \dots + b_1[\quad] + b_0[\quad]$$

$\downarrow$

$y = Cx + Du$

19.11.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \dots & \\ 0 & 0 & 1 & \dots & \\ & & \vdots & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & \dots & & & -a_{n-1} \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \qquad \dot{x} = Ax + Bu$$

$$y = C \cdot x + Du$$

$$y = \begin{bmatrix} b_0 & b_1 & \dots & b_m & 0 & \dots & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Zgled: $\dddot{y} + 3\ddot{y} + 4\dot{y} + y = 2\ddot{u} + 5\dot{u} + 7u$

$a_0 = 1 \qquad b_0 = 7$
$a_1 = 4 \qquad b_1 = 5$
$a_2 = 3 \qquad b_2 = 2$
$a_3 = 1$

$$\dddot{y} + 3\ddot{y} + 4\dot{y} + y = u$$

$$\left.\begin{aligned} x_1 &= y \\ x_2 &= \dot{x}_1 = \dot{y} \\ x_3 &= \dot{x}_2 = \ddot{y} \\ \dot{x}_3 &= \dddot{y} = \underbrace{-3\ddot{y} - 4\dot{y} - y + u} \\ \dot{x}_3 &= -x_1 - 4x_2 - 3x_3 + u \end{aligned}\right\} \quad \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

vstavimo $x_1 = y$ v zg. enačbo

$$\dddot{x}_1 + 3\ddot{x}_1 + 4\dot{x}_1 + x_1 = u \quad ①$$

najvišji odv. na desni strani je 2. odvod $\rightarrow$ 2x odv.

$$\frac{d}{dt}[\dddot{x}_1] + 3\frac{d}{dt}[\ddot{x}_1] + 4\frac{d}{dt}[\dot{x}_1] + \frac{d}{dt}[x_1] = \frac{d}{dt}[u] \quad \leftarrow x_2 = \dot{x}_1$$

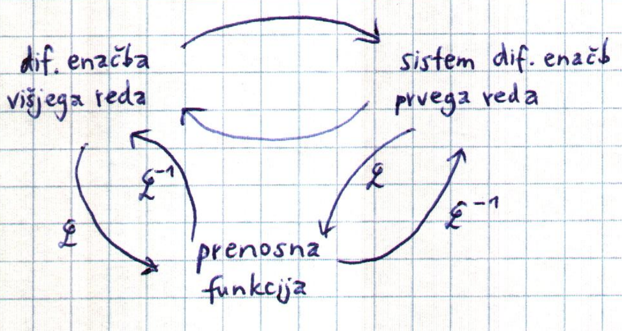
$$\dddot{x}_2 + 3\ddot{x}_2 + 4\dot{x}_2 + x_2 = \dot{u} \quad ②$$

$$\frac{d}{dt}[\dddot{x}_2] + 3\frac{d}{dt}[\ddot{x}_2] + 4\frac{d}{dt}[\dot{x}_2] + \frac{d}{dt}[x_2] = \ddot{u} \quad \leftarrow x_3 = \dot{x}_2$$

$$\dddot{x}_3 + 3\ddot{x}_3 + 4\dot{x}_3 + x_3 = \ddot{u} \quad ③$$

$$\dddot{y} + 3\ddot{y} + 4\dot{y} + y = 2[\dddot{x}_3 + 3\ddot{x}_3 + 4\dot{x}_3 + \underline{x_3}] + 5[\dddot{x}_2 + 3\ddot{x}_2 + 4\dot{x}_2 + \underline{x_2}] + 7[\dddot{x}_1 + 3\ddot{x}_1 + 4\dot{x}_1 + \underline{x_1}]$$

$$y = \begin{bmatrix} 7 & 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



Prenosna funkcija

$$u(t) \longrightarrow \boxed{\text{sistem}} \longrightarrow y(t)$$

$$\text{prenosna f.} = \frac{\mathcal{L}[\text{izhodni signal}]}{\mathcal{L}[\text{vhodni signal}]}$$

$$H(s) = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[u(t)]}$$

$$H(s) = \frac{Y(s)}{U(s)}$$

$$Y(s) = H(s) \cdot U(s)$$

$$h(t) = \mathcal{L}^{-1}\left[H(s) = \frac{Y(s)}{U(s)}\right]$$

$$\mathcal{L}^{-1}[Y(s) = H(s) \cdot U(s)]$$

$$y(t) = \int_0^t h(t-\tau) \cdot u(\tau)\, d\tau$$

spl. DE

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + \ldots + b_1 \frac{du}{dt} + b_0 u \quad / \mathcal{L}$$

$$y(0),\ \dot{y}(0), \ldots\ y^{(n-1)}(0)$$

$$s^n \cdot Y(s) + a_{n-1} \cdot s^{n-1} Y(s) + \ldots + a_1 s Y(s) + a_0 Y(s) = b_m \cdot s^m U[s] + \ldots + b_1 \cdot s \cdot U[s] + b_0 U[s]$$

$$Y[s]\,(s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0] = U[s]\,(b_m s^m + \ldots + b_1 s + b_0)$$

$$\boxed{H(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}}$$

$m < n$

matem. oblika je kvocient dveh polinomov

$P[s] = b_m s^m + \ldots$

$Q[s] = s^n + \ldots$ $\leftarrow$ karakteristični ali značilni polinom

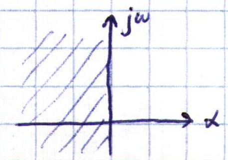
$Q[s] = 0 \rightarrow$ karakt. značilna enačba

$$\boxed{H[s] = \frac{P[s]}{Q[s]}}$$

$P[s] = 0$ $\quad z_1, z_2, \ldots\ z_m$ $\quad$ ničle

$Q[s] = 0$ $\quad p_1, p_2, \ldots\ p_n$ $\quad$ poli

$s_1, s_2, \ldots\ s_n$

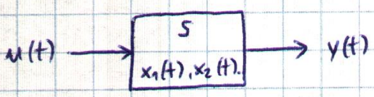


III. PROSTOR STANJ (state space)

III.1 Spremenljivke stanj

za zvezne, linearne, čas. nesprem. sisteme

$$\dot{x} = Ax + Bu$$



S pomočjo spremenljivk vhodnih signalov in matematičnega modela, ki opisuje dinamični sistem, lahko določimo stanja sistema in bodoče odzive.

$x_1(t), x_2(t), \dots x_n(t)$

$\dot{x} = Ax + Bu$ $\quad \boxed{\begin{matrix} A, B, u \\ x(0) \quad x(t_0) \end{matrix}}$

$x(t) \quad t \geq 0$
$\quad\quad\; t \geq t_0$

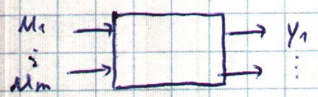
$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad t \geq t_0 \qquad x(t) = \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

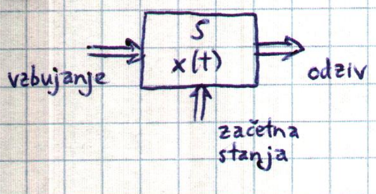
$y(t) = Cx + Du$
C, D, u, x
$y(t), \; t \geq t_0$

$$x^T = [\underbrace{x_1(t), x_2(t), \dots x_n(t)}]$$

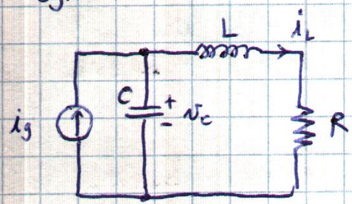
spremenljivke stanj (state variables)

Spremenljivke stanj so tiste spremenljivke, s katerimi bomo opisovali obnašanje sistema v prihodnosti. Poznati pa moramo matem. model (osnovni matriki A in B), vzbujanje u in začetna stanja spremenljivk stanj.





Zgled



$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix}$$

$\dot{x} = \dots ?$

? $\dot{x} = Ax + Bu$

① $i_g - i_C - i_L = 0$
$i_C = C \frac{dv_C}{dt}$

$$C \frac{dv_C}{dt} = -i_L + i_g$$

② $v_L + v_R - v_C = 0$
$v_L = \frac{di_L}{dt} \cdot L$

$$L \frac{di_L}{dt} = -R i_L + v_C$$

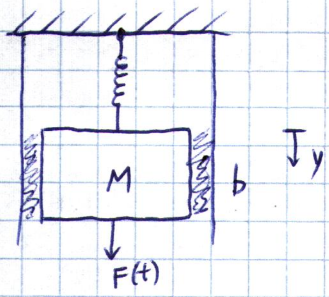
$$\frac{di_L}{dt} = -\frac{R}{L} \cdot i_L + \frac{1}{L} v_C$$

$$\frac{dv_C}{dt} = -\frac{1}{C} i_L + \frac{1}{C} i_g$$

$$\begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_C}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix} \cdot \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix} \cdot i_g$$

$i_L(0), v_C(0), i_g(t)$

A, B

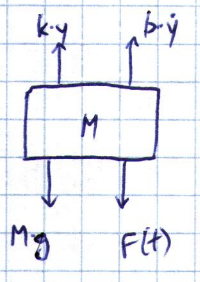


$$\sum F_i = M \cdot a$$

$$M\cdot g + F(t) - k\cdot y - b\dot{y} = M\cdot\ddot{y}$$

$$M\ddot{y} + b\dot{y} + ky = M\cdot g + F(t)$$

$$\dot{y} = v$$



$$M\cdot\dot{v} + b\cdot v + k\cdot y = M\cdot g + F(t)$$

$$\dot{v} = -\frac{k}{M}\cdot y - \frac{b}{M}\cdot v + g + \frac{1}{M}F(t)$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{b}{M} \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{M} \end{bmatrix} \cdot \begin{bmatrix} g \\ F(t) \end{bmatrix}$$

V splošnem lahko izberemo različne spremenljivke za spremenljivke stanj. Na splošno izbiramo direktno vodljive veličine.

$v_R,\ i_L$

$v_R - i_L$

$x_1^* = v_L \qquad x_1 = i_L$

$x_2^* = v_C \qquad x_2 = v_c$

$x_1^* = f_1(x_1, x_2)$

$x_2^* = f_2(x_1, x_2)$

$v_c = v_c$

$x_2^* = x_2$

$v_L = -R\, i_L + v_C$

$x_1^* \qquad x_1 \qquad x_2$

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \underbrace{\begin{bmatrix} -R & 1 \\ 0 & 1 \end{bmatrix}}_{T} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x^* = T\cdot x$$

$$x = T^{-1}\cdot x^*$$

$$\dot{x}^* = Ax^* + Bu$$

$$\dot{x} = Ax + Bu$$

$$T^{-1}\cdot\dot{x}^* = AT^{-1}x + Bu$$

$$\dot{x}^* = TAT^{-1}x + TBu$$

$$\dot{x} = Ax + Bu$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

III.2. Enačbe stanj sistema

Stanje sistema zapišemo s sistemom n-tih dif. enačb 1. reda.

$$\dot{x} = A\underline{x} + B\underline{u}$$
$$y = C\underline{x} + D\underline{u}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ \vdots \end{bmatrix}$$

$$\left.\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}u_1 + \ldots + b_{1n}u_m \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}u_1 + \ldots + b_{2m}u_m \\
&\ \vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_{n1}u_1 + \ldots + b_{nm}u_m
\end{aligned}\right\}\ \text{enačbe stanja sistema}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\text{vektor stanj sistema}} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ & & \vdots & \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \cdot \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}}_{\text{vektor vzbujanj/vhodnih signalov}}$$

enačbe stanja sistema v matrični obliki

Množico DE 1. reda imenujemo enačbe stanja sistema

A – osnovna matrika sistema (vedno kvadratna $A_{n\times n}$)
B – vhodna matrika ($B_{n\times m}$)

$$y(t) = \begin{bmatrix} y_1 \\ \vdots \\ y_\ell \end{bmatrix}$$

vektor odzivov

Odzivi so v splošnem različni od spremenljivk stanj.

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

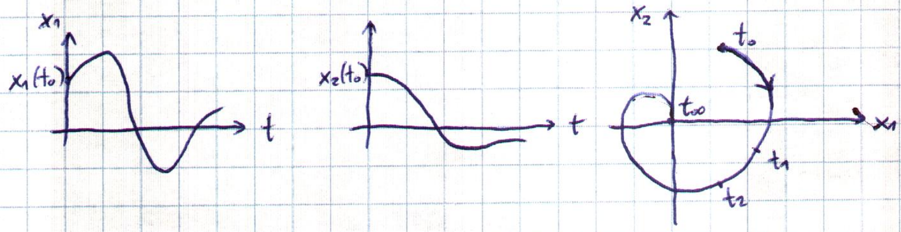
$$\begin{bmatrix} y_1 \\ \vdots \\ y_\ell \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & & \vdots & \\ c_{\ell 1} & c_{\ell 2} & \dots & c_{\ell n} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1m} \\ d_{21} & d_{22} & \dots & d_{2m} \\ & & & \\ d_{\ell 1} & d_{\ell 2} & \dots & d_{\ell m} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

C – izhodna matrika $C_{\ell\times n}$
D – neposredna matrika prehoda $D_{\ell\times m}$

III. 3. Prostor stanj

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \underset{\text{zač. stanje}}{x(0)} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \qquad x(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$
$$\downarrow$$
$x_1(t) = \dots$ $\quad t \geq t_0$
$x_2(t) = \dots$



Stanje sistema $x(t)$ v vsakem trenutku t lahko predstavimo s točko v prostoru. Ko čas t narašča od začetnega časa t_0 do t_∞, točka, ki predstavlja stanje, potuje po krivulji, ki jo imenujemo krivulja prostora stanj ali tirnica.
Točka potuje po prostoru, ki ga imenujemo prostor stanj, določajo ga pa spremenljivke stanj.

III. 4. Rešitev enačb stanj sistema

$$\dot{x} = Ax + Bu$$
$$\underline{x(t) = ?}$$
$$u(t) = 0$$
$$\boxed{\dot{x} = A \cdot x}$$
$$x(0)$$
$$\dot{x} = a \cdot x \qquad \frac{1}{x}\,dx = a \cdot dt \;/ \int$$
$$\frac{dx}{dt} = ax \qquad x = e^{at} + k$$

$$\boxed{x(t) = e^{At} \cdot x(0)} \quad t_0 = 0$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1n} \\ & & \vdots & \\ \phi_{n1} & \phi_{n2} & \dots & \phi_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}$$

$$e^{At} = \phi(t)$$

matrika prehajanja stanj

$\phi_{n\times n}$

govori o tem,
ϕ_{ij} – kakšen je doprinos j-tega začet. stanja k i-ti spremenljivki stanj

① $\Phi(t_0 - t_0) = I$ $\quad\quad$ $\Phi(t-t_0) = e^{A(t-t_0)}$

② $\Phi(t_2 - t_0) = \Phi(t_2 - t_1) \cdot \Phi(t_1 - t_0)$

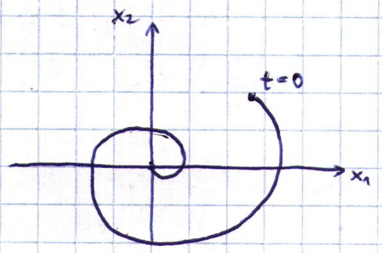
③ $\Phi(t_1 - t_0) = [\Phi(t_0 - t_1)]^{-1}$

$$\boxed{x(t) = e^{A(t-t_0)} \cdot x(t_0)}$$

če $t_0 = 0 \rightarrow x(t) = e^{At} \cdot x(0)$

26.11.

$\dot{x} = Ax + Bu$, $y = Cx + Du$



prostor stanj → prostor, ki ga določajo spremenljivke stanj

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \end{bmatrix} = ? \quad\quad t \geq 0$$

velja za linearne, časovno nespremenljive sisteme

prednost modeliranja sistema z diferencialnimi enačbami prvega reda je ta: ni odvodov začetnih stanj, zato so zač. stanja lažje določljiva

→ prednost, ki se nanaša na zač. stanja

$y(0)$, $y'(0)$

$x_1(0)$, $x_2(0)$

$\dot{x} = Ax \quad\quad u(t) = 0$

$\Downarrow$

$x(t) = e^{At} \cdot x(0)$

$$\boxed{x(t) = e^{A(t-t_0)} \cdot x(t_0)}$$

$\underset{n\times n}{\Phi}(t) = e^{At}$

1), 2), 3)

$\dot{x} = Ax + Bu \quad\quad x(t_0) \neq 0$

$u(t) \neq 0$

nehomogena matrična enačba stanj

$\dot{x} = Ax + Bu \;/\cdot\; e^{-At}$

$e^{-At} \cdot \dot{x} = e^{-At} Ax + e^{-At} Bu$

$e^{-At} \cdot x(t) \;/\; \frac{d}{dt}$

$\frac{d}{dt}(e^{-At} \cdot x(t)) = -e^{-At} \cdot A \cdot x(t) + e^{-At} \cdot \dot{x}(t)$

$e^{-At} \cdot \dot{x} = e^{-At} \cdot A \cdot x + \frac{d}{dt}(e^{-At} \cdot x)$

$d/dt\,(e^{-At} \cdot x) = e^{-At} \cdot B \cdot u \;/ \int$

$e^{-At} \cdot x = \int e^{-At} \cdot B \cdot u \, dt + K$

$x(t) = e^{At} \int e^{-At} \cdot B \cdot u \, dt + e^{At} \cdot K$

$x(t) = e^{At} \int^t e^{-A\tau} \cdot B \cdot u(\tau) \, d\tau + e^{At} \cdot K$

$x(t) = \int_0 e^{A(t-\tau)} \cdot B \cdot u(\tau) \, d\tau + e^{At} \cdot K$

$$\boxed{x(t) = \underbrace{e^{At} \cdot x(0)}_{\text{odziv na zač. stanje}} + \underbrace{\int_0^t e^{A(t-\tau)} \cdot B \cdot u(\tau) \, d\tau}_{\text{odziv na vzbujanje}}}$$

$K = x(0)$ $\quad$ (imamo vzbujanje + zač. stanje)

$$\boxed{x(t) = e^{A(t-t_0)} \cdot x(t_0) + \int_{t_0}^t e^{A(t-\tau)} B u(\tau) \, d\tau}$$

(splošna enačba)

analitično predstavlja ta integral problem, ker je treba integrirat vsak element matrike

III. 5. Določanje matrike prehajanja stanj e^{At}

$\phi(t) = ?$ $\quad t \geq 0$

$\dot{x} = Ax + Bu$ (numerično je ni problem rešit)

① e^{At} razvoj v vrsto

$$e^{at} = 1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \dots + \frac{(at)^n}{n!} + \dots$$

$a \to A$

$$e^{At} = I + A \cdot t + \frac{(At)^2}{2!} + \dots + \frac{(At)^n}{n!} + \dots$$

I – enotina matrika

$$e^{At} \approx I + A \cdot t + \frac{(At)^2}{2!}$$

na roke ne moremo poračunat neskončno št. členov, zato vzamemo končno št. členov

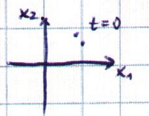
1. primer: $A = \begin{bmatrix} -2 & -1 \\ 2 & 0 \end{bmatrix}$

$$e^{At} \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2t & -t \\ 2t & 0 \end{bmatrix} + \begin{bmatrix} t^2 & t^2 \\ -2t^2 & -t^2 \end{bmatrix}$$

$$e^{At} \approx \begin{bmatrix} 1-2t+t^2 & -t+t^2 \\ 2t-2t^2 & 1-t^2 \end{bmatrix}$$

slabost:
1.) od števila členov je odvisna natančnost
2.) iz elementov matrik ne moremo določit, kam vrsta konvergira

– je nenatančna



(napaka se kvadrira)

② Laplace (reševanje z Lapl. presl.)

$$\begin{tikzcd} t \arrow[r, "\mathcal{L}"] & s \arrow[d] \\ t \arrow[u, dotted] & s \arrow[l, "\mathcal{L}^{-1}"] \end{tikzcd}$$

$$\dot{x} = Ax + Bu \quad / \mathcal{L}$$

$$sX(s) - x(0) = A \cdot X(s) + B \cdot U(s)$$

$$X(s) = ?$$

$$sX(s) - A \cdot X(s) = x(0) + B \cdot U(s)$$

$$(sI - A) \cdot X(s) = X(0) + B \cdot U(s) \quad / \cdot (sI-A)^{-1}$$

$$X(s) = (sI-A)^{-1} \cdot x(0) + (sI-A)^{-1} \cdot B \cdot U(s)$$

$$x(t) = \mathcal{L}^{-1}[X(s)]$$

$$x(t) = \mathcal{L}^{-1}\left[(sI-A)^{-1} \cdot x(0)\right] + \mathcal{L}^{-1}\left[(sI-A)^{-1} \cdot B \cdot U(s)\right]$$

$$\boxed{e^{At} = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right]}$$

2. način določanja matrike prehajanja stanj s pomočjo (inverzne) Lapl. trans.

to rešitev primerjamo s tisto, ki smo jo že izpeljali

$$x(t) = e^{At} \cdot x(0) + \int_0^t e^{A(t-\tau)} \cdot B \cdot u(\tau)\, d\tau$$

2. primer: $A = \begin{bmatrix} -2 & -1 \\ 2 & 0 \end{bmatrix}$ $\quad e^{At} = ?$

$$(sI - A) = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ 2 & 0 \end{bmatrix}$$

$$(sI - A) = \begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix} \quad \begin{bmatrix} \phi_{11}(s) & \phi_{12}(s) \\ \phi_{21}(s) & \phi_{22}(s) \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{\det}\begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix} =$$

$$\det = s^2 + 2s + 2$$

treba je poiskat inverzno Lap. transf. za vsak element matrike posebej

$\Phi_{11}(t) = \mathcal{L}^{-1}[\Phi_{11}(s)]$

$\vdots$

$$\Phi_{11} = \frac{s}{s^2+2s+2} = \frac{\sim}{(s-s_1)(s-s_2)} = \frac{\sim^2}{s-s_1} + \frac{\sim}{s-s_2}$$

$$\Phi_{11} = \frac{(s+1)-1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

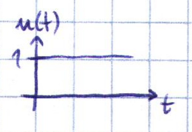
$$\Phi_{11}(t) = \mathcal{L}^{-1}[\Phi_{11}(s)] = e^{-t}\cdot \cos t - e^{-t}\cdot \sin t$$

$\Phi_{12} = \dots$

$$e^{At} = e^{-t}\begin{bmatrix} \cos t - \sin t & -\sin t \\ 2\cdot \sin t & \cos t + \sin t \end{bmatrix} \qquad t \geq 0$$

$\dot{x} = Ax + Bu$

$A = \begin{bmatrix} -2 & -1 \\ 2 & 0 \end{bmatrix} \quad x(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad u(t) \dots$ enotina stopnica



$x(t) = ?$

$$x(t) = e^{At}\cdot x(0) + \int_0^t e^{A(t-\tau)}\cdot B\cdot u(\tau)\,d\tau = x_z + x_v$$

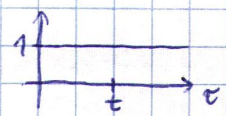
odziv na vzbujanje

odziv na zač. stanje

$$x_z(t) = 2\cdot e^{-t}\begin{bmatrix} \cos t - \sin t \\ 2\sin t \end{bmatrix}$$

$$e^{A(t-\tau)} = e^{-(t-\tau)}\begin{bmatrix} \cos(t-\tau) - \sin(t-\tau) & \dots \\ 2\sin(t-\tau) & \dots \end{bmatrix}$$

$$e^{A(t-\tau)}\cdot B = e^{-(t-\tau)}\begin{bmatrix} \cos(t-\tau) - \sin(t-\tau) \\ 2\sin(t-\tau) \end{bmatrix}$$



$$\int_0^t e^{-(t-\tau)}\begin{bmatrix} \cos(t-\tau) - \sin(t-\tau) \\ 2\sin(t-\tau) \end{bmatrix} d\tau$$

$t - \tau = z$

$-d\tau = dz$

$$x(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\cdot x(0)\right] + \mathcal{L}^{-1}\left[(sI-A)^{-1}\cdot B\cdot U(s)\right]$$

enotina stopnica!

$U(s) = \frac{1}{s}$

$$\frac{1}{s^2+2s+1}\begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cdot\frac{1}{s}$$

$$\frac{1}{s^2+2s+1}\begin{bmatrix} s \\ 2 \end{bmatrix}\cdot\frac{1}{s} = \frac{1}{s(s+s_1)(s+s_2)} = \frac{\sim}{s} + \frac{\sim}{s+s_1} + \frac{\sim}{s+s_2}$$

$e^{At} = ?$

③ Določanje matrike prehodnih stanj e^{At} s pomočjo lastnih vrednosti matrike A

$A\cdot u = \lambda\cdot u$

$u \dots$ vektor (lastni vektor matrike A)

$\lambda \dots$ skalar (lastna vrednost matrike A)

$A \dots$ matrika (kvadratna)

$u \dots$ l. v. m. A

$\lambda \dots$ -ll-

$$A u - \lambda \cdot u = 0$$
$$(A - \lambda I) \cdot u = 0$$
$$|A - \lambda I| = 0$$
↑ determinanta !

$$g(\lambda) = |A - \lambda I|$$
↖ značilni karakteristični polinom

$$g(\lambda) = 0$$ ← značilna oz. karakteristična enačba

↳ λ_1, λ_2

$$g(\lambda) = (\lambda - \lambda_1) \cdot (\lambda - \lambda_2) \ldots$$

$\lambda_i \rightarrow u_i$ (vsaki lastni vrednosti pripada l. vektor)

$$A \cdot u_i = \lambda_i \cdot u_i$$

Primer:

$$A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \qquad \begin{matrix} \lambda_1, \lambda_2 \\ u_1, u_2 \end{matrix}$$

$$|A - \lambda I| = \begin{vmatrix} -3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = 0$$

$$g(\lambda) = \lambda^2 + 3\lambda + 2 = 0$$
$$g(\lambda) = (\lambda + 1)(\lambda + 2) = 0$$
$$\boxed{\begin{matrix} \lambda_1 = -1 \\ \lambda_2 = -2 \end{matrix}}$$

lastne vrednosti so v splošnem kompleksne (so koreni v kompl. ravn.)

Pri enačbah ~~nižjega~~ višjega reda stabilnost določamo s koreni imenovalca dif. enačbe, pri enačbah prvega reda pa z Nebulasovo krivuljo.

Koreni imenovalca določajo stabilnost sistema

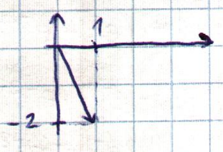
$$s_1, s_2$$
$$\lambda_1 = s_1$$
$$\lambda_2 = s_2$$

1. lastni vektor:

$$A \cdot \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \lambda_1 \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = - \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix}$$

$$\left.\begin{matrix} -3 u_{11} - u_{12} = - u_{11} \\ 2 \cdot u_{11} = - u_{12} \end{matrix}\right\} \begin{matrix} u_{11} = 1 \\ u_{12} = -2 \end{matrix}$$



2. lastni vektor:

$$\begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = -2 \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}$$

$$u_{21} = 1$$
$$u_{22} = -1$$

$$u_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A \Rightarrow \lambda_1, \lambda_2 \rightarrow e^{At} = ?$$

upoštevamo teorem: Cayley-Hamiltonov teorem → vsaka kvadratna matrika zadosti svoji značilni oz. karakteristični enačbi

$$g(\lambda) = |A - \lambda I| = 0$$
$$\lambda \rightarrow A$$
$$g(A) = |A - AI| = |A - A| = 0$$
$$g(\lambda) = 0$$
$$g(A) = 0$$

predpostavimo, da je značilni polinom, polinom prve stopnje:

$$g(\lambda) = \underset{\substack{\uparrow \\ A_{n\times n}}}{a_n} \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$$

$$g(A) = a_n A^n + a_{n-1} A^{n-1} + \ldots + a_1 A + a_0 I = 0$$

$$A^n = -\frac{a_{n-1}}{a_n} A^{n-1} - \frac{a_{n-2}}{a_n} A^{n-2} - \ldots - \frac{a_1}{a_n} A - \frac{a_0}{a_n} I$$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \ldots + \frac{(At)^n}{n!} + \ldots$$

$$A^{n+1} = A^n \cdot A$$

$\vdots$

s potencami: $I, A, \ldots, A^{n-1}$

∞

$$\boxed{e^{At} = \sum_{k=0}^{n-1} \alpha_k A^k} \qquad \alpha_0, \alpha_1, \ldots, \alpha_{n-1} = ?$$

Kako določimo α?

$$\boxed{e^{\lambda t} = \sum_{k=0}^{n-1} \alpha_k \lambda^k \rightarrow \alpha_0, \alpha_1, \ldots, \alpha_{n-1}}$$

$A_{n\times n}$ $\qquad e^{At}_{n\times n} = ?$

1. $A \Rightarrow \lambda_i$
2. $e^{\lambda t} = \sum_{k=0}^{n-1} \alpha_k \lambda^k \leftarrow \lambda_0, \lambda_1, \ldots$
3. $\alpha_0, \alpha_1, \ldots$
4. $e^{At} = \sum_{k=0}^{n-1} \alpha_k A^k$

λ - v splošnem kompleksne!

Primer:

a) $\lambda \ldots$ realne (vse lastne vrednosti so realne)

$$A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}$$

$$g(\lambda) = |A - \lambda I| = \begin{vmatrix} -3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = 0$$

$$g(\lambda) = \lambda^2 + 3\lambda + 2 = 0$$

$$g(\lambda) = (\lambda+1)(\lambda+2) = 0$$

$$\lambda_1 = -1$$

$$\lambda_2 = -2$$

$n=2$

$$e^{At} = \sum_{k=0}^{1} \alpha_k A^k$$

$$e^{At} = \alpha_0 \cdot I + \alpha_1 \cdot A$$

$$e^{\lambda t} = \alpha_0 + \alpha_1 \lambda \Rightarrow \alpha_0, \alpha_1$$

$$\left.\begin{array}{ll} \lambda = -1 & e^{-t} = \alpha_0 - \alpha_1 \\ \lambda = -2 & e^{-2t} = \alpha_0 - 2\alpha_1 \end{array}\right\} -$$

$\alpha_0, \alpha_1 = ?$

$$e^{-t} - e^{-2t} = -\alpha_1 + 2\alpha_1$$

$$\alpha_1 = e^{-t} - e^{-2t}$$

$$\alpha_0 = 2e^{-t} - e^{-2t}$$

$$e^{At} = \alpha_0 I + \alpha_1 A =$$

$$= (2e^{-t} - e^{-2t}) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{-t} - e^{-2t}) \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} -e^{-t} + 2e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & 2e^{-t} - e^{-2t} \end{bmatrix}$$

$$\Uparrow e^{At} \cdot x(0)$$

lastne vrednosti so zelo pomembne $\rightarrow$ kako se bo sistem obnašal

Primer: (kompleksne lastne vrednosti)

$$A = \begin{bmatrix} -2 & -1 \\ 2 & 0 \end{bmatrix}$$

$$g(\lambda) = |A - \lambda I| = \begin{vmatrix} -2-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = 0$$

$$g(\lambda) = \lambda^2 + 2\lambda + 2 = 0$$

$$\lambda_1 = -1 + j$$

$$\lambda_2 = -1 - j$$

$$e^{At} = \alpha_0 I + \alpha_1 A \qquad \alpha_0, \alpha_1 = ?$$

$$e^{\lambda t} = \alpha_0 + \alpha_1 \lambda$$

$$\left.\begin{aligned} \lambda_1:\ e^{(-1+j)t} &= \alpha_0 + \alpha_1(-1+j) \\ \lambda_2:\ e^{(-1-j)t} &= \alpha_0 + \alpha_1(-1-j) \end{aligned}\right\} -$$

$$e^{(-1+j)t} - e^{(-1-j)t} = \alpha_1(-1+j) - \alpha_1(-1-j)$$

$$e^{-t}(e^{jt} - e^{-jt}) = 2j\alpha_1$$

$$\alpha_1 = e^{-t}\,\frac{e^{jt} - e^{-jt}}{2j} = e^{-t} \cdot \sin t$$

$$\alpha_0 = e^{-t}(\cos t + \sin t)$$

$$e^{At} = \alpha_0 I + \alpha_1 A$$

$$e^{At} = e^{-t}\begin{bmatrix} -\sin t + \cos t & -\sin t \\ 2\sin t & \sin t + \cos t \end{bmatrix}$$

Primer:

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$g(\lambda) = |A - \lambda I| = \begin{vmatrix} -2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$g(\lambda) = \lambda^2 + 2\lambda + 1 = 0$$

$$g(\lambda) = (\lambda + 1)^2 = 0$$

$$\lambda_1 = \lambda_2 = -1 \quad \longrightarrow \quad e^{\lambda t} = \alpha_0 + \alpha_1 \lambda$$

$$e^{\lambda t} = \alpha_0 + \alpha_1 \lambda \quad \Big/ \frac{d}{d\lambda}$$

$$t \cdot e^{\lambda t} = \alpha_1$$

$$\alpha_1 = t \cdot e^{-t}$$

$$e^{-t} = \alpha_0 - \alpha_1$$

$$e^{-t} = \alpha_0 - t e^{-t}$$

$$\alpha_0 = e^{-t} + t \cdot e^{-t}$$

$$e^{At} = \alpha_0 I + \alpha_1 A$$

$$\dot{x} = Ax + Bu \Rightarrow x(t)$$
$$y = Cx + Du$$

3.12.

$$y(t)$$
$$x(t) = e^{At} \cdot x(0) + \int_0^t e^{A(t-\tau)} \cdot B \cdot u(\tau)\, d\tau$$

e^{At} $\quad A_{n \times n}$

① $e^{At} = I + \ldots$
② $\mathcal{L}$
③ λ lastne vred.
④ λ, u

Določanje matrike preh. stanj e^{At} s pomočjo lastnih vektorjev matrike A (Diagonalizacija osnovne matrike A)

$$A = \Lambda$$

bolj redka matrika → lažje določit e^{At}

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$g(\lambda) = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0$$

$$g(\lambda) = (2-\lambda)(3-\lambda) = 0$$

$\lambda_1 = 2$
$\lambda_2 = 3$

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad e^{At} = \begin{bmatrix} e^{x_1 t} & 0 \\ 0 & e^{x_2 t} \end{bmatrix}$$

splošno: $$e^{At} = \begin{bmatrix} e^{x_1 t} & 0 & \cdots & \\ 0 & e^{x_2 t} & & \\ \vdots & & \ddots & \\ & & & e^{x_n t} \end{bmatrix}$$

Iskanje matrike preh. stanj je v primeru diagonalne matrike A zelo enostavna. V splošnem ne bo diag., zato jo diagonaliziramo.

$$\begin{tikzcd} A \arrow[r, "\text{diagonalizacija}"] \arrow[d, dotted, "①②③"'] & D \arrow[d] \\ e^{At} & e^{Dt} \arrow[l] \end{tikzcd}$$

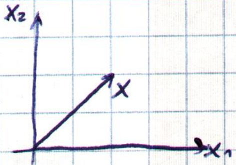
$D = \Lambda$
$A \Rightarrow D$ $\quad$ včasih ne bo diag., ampak najbolj redka matrika
↳ če se ne da

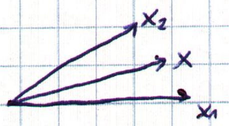
$u \ldots$ lastni vektorji

$$Au = \lambda u$$
$$Au_1 = \lambda_1 u_1$$
$$\vdots$$

2

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$





$$\Lambda = U^{-1} \cdot A \cdot U$$

$$\boxed{\Lambda = \Theta^{-1} \cdot A \cdot \Theta}$$

$$\Theta = \left[u_1 \vdots u_2 \vdots \ldots \vdots u_n \right]$$ lastni vektorji

$$A \rightarrow \lambda_1, \lambda_2 \ldots \lambda_n$$

$$\Downarrow$$

$$u_1, u_2, \ldots u_n$$

$$\Theta = \left[u_1 \vdots u_2 \vdots \ldots \vdots u_n \right]$$

$$\Theta^{-1}$$

$$\Lambda = \Theta^{-1} \cdot A \cdot \Theta$$

$$\Downarrow$$

$$e^{\Lambda t}$$

$$e^{At} = \Theta \cdot e^{\Lambda t} \cdot \Theta^{-1}$$

$$A = \Theta \cdot \Lambda \cdot \Theta^{-1}$$

Določanje lastnih vektorjev v primeru večkratnih lastnih vrednosti

Zgled: $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$

$$g(\lambda) = |A - \lambda I| = \begin{vmatrix} 3-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$g(\lambda) = \lambda^2 - 4\lambda + 3 - 8 = 0$$

$$g(\lambda) = (\lambda - 5)(\lambda + 1) = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = 5 \qquad \lambda_1 \neq \lambda_2$$

$$\Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{5t} \end{bmatrix}$$

$$e^{At} = \Theta \cdot e^{\Lambda t} \cdot \Theta^{-1}$$

$$\Theta = \left[u_1 \vdots u_2 \right]$$

$$A \cdot u = \lambda u$$

$$A \cdot u_1 = \lambda_1 u_1$$

$$A \cdot u_2 = \lambda_2 u_2$$

$$\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = - \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}$$

$$3u_{11} + 4u_{21} = -u_{11}$$

$$4u_{11} = -4u_{21}$$

$$u_{11} = -u_{21}$$

$$u_{11} = 1, \quad u_{21} = -1$$

$$u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = 5 \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix}$$

$$3u_{21} + 4u_{22} = 5u_{21}$$

$$4u_{22} = 2u_{21}$$

$$2u_{22} = u_{21}$$

$$u_{22} = 1, \quad u_{21} = 2$$

$$u_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Theta = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\Theta^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

želimo dokazat: $e^{At} = \Theta \cdot e^{\Lambda t} \Theta^{-1}$:

izraza sta enaka: dokaz ✓

$$e^{At} = \sum_{k=0}^{n-1} \alpha_k A^k$$

$A \to \Lambda$

$$e^{\Lambda t} = \sum_{k=0}^{n-1} \alpha_k \Lambda^k$$

$$e^{At} = \Theta \cdot e^{\Lambda t} \cdot \Theta^{-1} = \Theta \sum_{k=0}^{n-1} \alpha_k \Lambda^k \cdot \Theta^{-1} = \sum_{k=0}^{n-1} \alpha_k \Theta \Lambda^k \Theta^{-1}$$

$$\Lambda = \Theta^{-1} \cdot A \cdot \Theta$$

$$\Lambda^2 = \Lambda \cdot \Lambda = \Theta^{-1} \cdot A \cdot \underbrace{\Theta \cdot \Theta^{-1}}_{I} \cdot A \cdot \Theta$$

$$\Lambda^2 = \Theta^{-1} \cdot A^2 \cdot \Theta$$

$$\Lambda^k = \Theta^{-1} \cdot A^k \cdot \Theta$$

$$e^{At} = \sum_{k=0}^{n-1} \alpha_k \underbrace{\Theta \cdot \Theta^{-1}} \cdot A^k \cdot \underbrace{\Theta \cdot \Theta^{-1}}$$

$$e^{At} = \sum_{k=0}^{n-1} \alpha_k A^k$$

$$A \Rightarrow \lambda_i \Rightarrow \mu_i \Rightarrow \Theta, \Theta^{-1}, e^{\Lambda t} \Rightarrow e^{At} = \Theta \cdot e^{\Lambda t} \cdot \Theta^{-1}$$

$A_{n \times n}$

$$g(\lambda) = |A - \lambda I| = 0$$

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

$$\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_n$$

$$f(\lambda, \mu) = \frac{g(\lambda) - g(\mu)}{\lambda - \mu}$$

$$g(\lambda) = |A - \lambda I| = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0$$

$$g(\mu) = a_n \mu^n + a_{n-1} \mu^{n-1} + \ldots + a_1 \mu + a_0$$

$$f(\lambda, \mu) = \frac{(a_n \lambda^n + \ldots + a_1 \lambda + a_0) - (a_n \mu^n + \ldots + a_1 \mu + a_0)}{\lambda - \mu}$$

$$f(\lambda, \mu) = \frac{a_n(\lambda^n - \mu^n) + \ldots + a_1(\lambda - \mu)}{\lambda - \mu}$$

$$f(\lambda, \mu) = a_n \frac{\lambda^n - \mu^n}{\lambda - \mu} + a_{n-1} \frac{\lambda^{n-1} - \mu^{n-1}}{\lambda - \mu} + \ldots + a_1 \frac{\lambda - \mu}{\lambda - \mu}$$

↑ značilnost teh parcialnih ulomkov je, da se vsak števec da deliti z imenovalcem brez ostanka

$\lambda \to \lambda I$

$\mu \to A$

$$f(\lambda I, A) = (\lambda I - A)^{-1} \cdot (g(\lambda I) - g(A))$$

matrik se ne da delit

$g(\lambda) = 0$

$g(A) = 0$ Cayley - Hamilton

$$f(\lambda I, A) = (\lambda I - A)^{-1} \cdot g(\lambda I) \quad / \; (\lambda I - A)$$

$$(\lambda I - A) \cdot f(\lambda I, A) = g(\lambda I)$$

$$g(\lambda I) = (\lambda I - A) \cdot \underbrace{f(\lambda I, A)}_{C(\lambda)}$$

$$g(\lambda I) = g(\lambda) \cdot I$$

$$g(\lambda)\cdot I = (\lambda I - A)\cdot C(\lambda)$$
$$\lambda \to \lambda_i$$
$$g(\lambda_i)\cdot I = (\lambda_i I - A)\cdot C(\lambda_i)$$

$g(\lambda) = a_n\lambda^n + \ldots + a_1\lambda + a_0 = 0$ (značilni polinom)
$$g(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\ldots(\lambda-\lambda_n)$$
$g(\lambda_i) = 0 \quad \forall i$
(za vsak i)

$$(\lambda_i I - A)\cdot C(\lambda_i) = 0$$
$$A\cdot u = \lambda \cdot u$$
$$\lambda\cdot u - A\cdot u = 0$$
$(\lambda I - A)\cdot u = 0$
$(\lambda_i I - A)\cdot C(\lambda_i) = 0$

s primerjavo teh dveh enačb vidimo, da se lastni vektorji skrivajo v $C(\lambda_i)$

$$\boxed{C(\lambda_i) \Rightarrow u_i}$$

$C(\lambda)$
$C(\lambda_1) \to u_1$
$C(\lambda_2) \to u_2$
$\vdots$

Ker vsaki lastni vrednosti pripada samo en lastni vektor, so stolpci matrike $C(\lambda_i)$ izluščimo le en lastni vektor. Katerikoli stolpec matrike $C(\lambda_i)$ lahko izberemo za lastni vektor u_i.

Primer: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow u_1, u_2$

$$g(\lambda) = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$g(\lambda) = \lambda^2 - 4\lambda + \overbrace{4-1}^{3} = 0$$
$$g(\lambda) = (\lambda-1)(\lambda-3) = 0$$
$$\lambda_1 = 1,\ \lambda_2 = 3$$

$$f(\lambda,\mu) = \frac{g(\lambda)-g(\mu)}{\lambda-\mu}$$
$$f(\lambda,\mu) = \frac{(\lambda^2-4\lambda+3)-(\mu^2-4\mu+3)}{\lambda-\mu} = \frac{\lambda^2-\mu^2}{\lambda-\mu} - 4\,\frac{\lambda-\mu}{\lambda-\mu}$$

$$\lambda^2-\mu^2 = (\lambda+\mu)(\lambda-\mu)$$

$$f(\lambda I, A) = C(\lambda)$$
$$C(\lambda) = \lambda I + A - 4I$$
$$C(\lambda) = A + (\lambda-4)I$$

$$C(\lambda) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} \lambda-4 & 0 \\ 0 & \lambda-4 \end{bmatrix}$$

$$C(\lambda) = \begin{bmatrix} \lambda-2 & 1 \\ 1 & \lambda-2 \end{bmatrix}$$

$$C(\lambda=\lambda_1) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$C(\lambda=\lambda_2) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Theta = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$
$$\Theta^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix},\quad e^{\Lambda t} = \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix},\quad e^{At} = \Theta\, e^{\Lambda t}\cdot \Theta^{-1}$$

$A_{n\times n} \Rightarrow g(\lambda)=0$

$\lambda_1, \lambda_2, \dots, \lambda_i, \dots$ m-kratna l.v.

$g(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\dots\underbrace{(\lambda-\lambda_i)^m}_{m\dots \text{ lastnih vektorjev}}\dots$

$A \to g(\lambda) \to \lambda$

$$f(\lambda,\mu) = \frac{g(\lambda)-g(\mu)}{\lambda-\mu}$$

$\lambda \to \lambda I$

$\mu \to A$

$f(\lambda I, A) = C(\lambda)$

$g(\lambda)\cdot I = (\lambda I - A)\cdot C(\lambda)$

ničta enačba

$(\lambda I - A)\cdot C(\lambda) = g(\lambda)\cdot I \quad 0 \quad / \frac{d}{d\lambda}$ m-1 odvajamo

$(\lambda I - A)\cdot C'(\lambda) + C(\lambda) = g'(\lambda)\cdot I \quad 1 \quad / \frac{d}{d\lambda}$

$(\lambda I - A)\cdot C''(\lambda) + C'(\lambda) + C'(\lambda) = g''(\lambda) I$

$(\lambda I - A)\cdot C''(\lambda) + 2C'(\lambda) = g''(\lambda) I \quad 2$

$$\left.\begin{array}{ll} 0 & (\lambda I - A)\cdot C(\lambda) = g(\lambda)\cdot I \\ 1. & (\lambda I - A)\cdot C'(\lambda) + C(\lambda) = g'(\lambda) I \\ 2. & (\lambda I - A)\cdot C''(\lambda) + 2C'(\lambda) = g''(\lambda) I \\ & \vdots \\ m-1 & (\lambda I - A)\, C^{(m-1)}(\lambda) + (m-1)\cdot C^{(m-2)}(\lambda) = g^{(m-1)}(\lambda)\cdot I \end{array}\right\} \text{dobimo m-enačb}$$

$\lambda \to \lambda_i$

$$\left.\begin{array}{l} g(\lambda_i) = 0 \\ g'(\lambda_i) = 0 \\ \vdots \\ g^{(m-1)}(\lambda_i) = 0 \end{array}\right\} g^{(j-1)}(\lambda_i) = 0 \; ; \quad j = 1,2,\dots,m$$

$(\lambda_i I - A)\cdot C(\lambda_i) = 0$

$(\lambda_i I - A)\cdot C'(\lambda_i) + C(\lambda_i) = 0$

$(\lambda_i I - A)\cdot C''(\lambda_i) + 2C'(\lambda_i) = 0$

$\vdots$

$(\lambda_i I - A)\cdot C^{(m-1)}(\lambda_i) + (m-1) C^{(m-1)}(\lambda_i) = 0$

$(A - \lambda_i I)\cdot C(\lambda_i) = 0$

$(A - \lambda_i I)\cdot C'(\lambda_i) = C(\lambda_i)$

$\vdots$

$\underline{(A - \lambda_i I)\cdot C^{(m-1)}(\lambda_i) = (m-1)\, C^{(m-2)}(\lambda_i)}$

$$\text{m enačb}\left\{\begin{array}{l} (A - \lambda_i I)\cdot C(\lambda_i) = 0 \\ (A - \lambda_i I)^2\cdot C'(\lambda_i) = 0 \\ \vdots \\ (A - \lambda_i I)^m\cdot C^{(m-1)}(\lambda_i) = 0 \end{array}\right\} \Longleftrightarrow \begin{array}{l} A\cdot u = \lambda\cdot u \\ (A-\lambda I)\cdot u = 0 \\ (A-\lambda I)^2\cdot u = 0 \\ \vdots \end{array}$$

$\underbrace{C(\lambda_i), C'(\lambda_i), \dots, C^{(m-1)}(\lambda_i)}$ tukaj se skriva u_{im} lastnih vektorjev

$u_{i1}, u_{i2}, \dots, u_{im}$

V primeru m-kratne lastne vrednosti moramo zadostiti m-tim enačbam. M-enačb da m-lastnih vektorjev, ki jih imenujemo posplošeni lastni vektorji.

Postopek: poiščemo $C(\lambda)$, vstavimo λ_i in dobimo $C(\lambda_i)$, iz $C(\lambda_i)$ izluščimo čim več linearno neodvisnih lastnih vektorjev (lin. neod. stolpci); če nismo uspeli določiti vseh m-lastnih vektorjev, določimo matriko $C'(\lambda)$, vstavimo λ_i in postopek za $C'(\lambda_i)$ ponovimo.

Moramo biti previdni, ker so lahko nekateri lastni vektorji izločeni iz matrike $C^{(k)}(\lambda_i)$ izločeni že iz prejšnjih matrik (moramo pogledat, da stolpec ni lin. kombinacija obstoječih l.v.)

Primer:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$g(\lambda) = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$g(\lambda) = \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$$g(\lambda) = (\lambda-5)(\lambda-1)^2 = 0$$

$$\lambda_1 = 5$$

$$\lambda_2 = \lambda_3 = 1$$

$$f(\lambda,\mu) = \frac{g(\lambda)-g(\mu)}{\lambda-\mu} = \frac{(\lambda^3 - 7\lambda^2 + 11\lambda - 5) - (\mu^3 - 7\mu^2 + 11\mu - 5)}{\lambda - \mu}$$

$$f(\lambda,\mu) = \frac{\lambda^3-\mu^3}{\lambda-\mu} - 7\,\frac{\lambda^2-\mu^2}{\lambda-\mu} + 11\,\frac{\lambda-\mu}{\lambda-\mu}$$

$$f(\lambda,\mu) = (\lambda^2 + \lambda\mu + \mu^2) - 7(\lambda+\mu) + 11$$

$$\lambda \to \lambda I$$

$$\mu \to A$$

$$C(\lambda) = (\lambda^2 - 7\lambda + 11) I + A^2 + (\lambda - 7)\cdot A$$

$$C(\lambda) = \begin{bmatrix} \lambda^2-5\lambda+4 & 2\lambda-2 & \lambda-1 \\ \lambda-1 & \lambda^2-4\lambda+3 & \lambda-1 \\ \lambda-1 & 2\lambda-2 & \lambda^2-5\lambda+4 \end{bmatrix}$$

$$C(\lambda=\lambda_1) = \begin{bmatrix} 4 & 8 & 4 \\ 4 & 8 & 4 \\ 4 & 8 & 4 \end{bmatrix} \rightarrow u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C(\lambda=\lambda_2=\lambda_3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C'(\lambda) = \begin{bmatrix} 2\lambda-5 & 2 & 1 \\ 1 & 2\lambda-4 & 1 \\ 1 & 2 & 2\lambda-5 \end{bmatrix}$$

$$C'(\lambda=\lambda_1=\lambda_2) = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \rightarrow u_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

(treba je preveriti, če sta linearna kombinacija obstoječih)

$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} \cdot B \cdot u(\tau)\, d\tau$

$y(t) = Cx + Du$

$e^{At} = ?$

e^{At}

$A \to J \to e^{Jt} \to e^{At}$ $\nearrow \Lambda$

$\uparrow$

Θ, Θ^{-1} $\qquad J = \Theta^{-1} \cdot A \cdot \Theta$

$\Theta = [u_1 \vdots \dots \vdots u_n]$

$A \cdot u_i = \lambda_i u_i \to u_i$

$\lambda_i = \lambda_i$

$\lambda_i \neq \lambda_j$

$J = \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$e^{Jt} = e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$

$\dots (\lambda - \lambda_j)^m \dots$

Zgled: $A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$, $u_1, u_2 = ?$

$g(\lambda) = |A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0$

$g(\lambda) = \lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$

$\lambda_1 = \lambda_2 = 2$

$f(\lambda, \mu) = \frac{g(\lambda) - g(\mu)}{\lambda - \mu} = \frac{(\lambda^2 - 4\lambda + 4) - (\mu^2 - 4\mu + 4)}{\lambda - \mu} = \frac{(\lambda^2 - \mu^2) - 4(\lambda - \mu)}{\lambda - \mu} =$

$= \lambda + \mu - 4 = (\lambda - 4) + \mu$

$\lambda \to \lambda I, \ \mu \to A$

$f(\lambda I, A) = C(\lambda) = (\lambda - 4) I + A$

$C(\lambda) = \begin{bmatrix} \lambda - 1 & 1 \\ -1 & \lambda - 3 \end{bmatrix}$ $\qquad C'(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad u_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

u_1, u_2

$C(\lambda), C'(\lambda)$

3 možn.: $C(\lambda) \to u_1, u_2$

$C(\lambda) \to u_1, \ C'(\lambda) \to u_2$

$C'(\lambda) \to u_1, u_2$

$C(\lambda = 2) = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

$u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

razlika med primeroma ?

① oba lastna vekt. smo določili iz matrike $C'(\lambda) \to u_2, u_3$

② $C(\lambda) \to u_1$

$C'(\lambda) \to u_2$

vpliva na $\underbrace{J}_{\text{Jordanova kanonična forma}} = \Theta^{-1} \cdot A \cdot \Theta$?

① $J = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \Lambda$

$$e^{Jt} = \begin{bmatrix} e^{5t} & & 0 \\ & e^{t} & \\ 0 & & e^{t} \end{bmatrix} \qquad e^{At} = \Theta \cdot e^{Jt} \cdot \Theta^{-1}$$

② $\Theta = [u_1 \vdots u_2] = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

$$\Theta^{-1} = \frac{1}{1} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$J = \Theta^{-1} \cdot A \cdot \Theta = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Če lastne v. določimo iz ene matrike, ~~bo~~ J diagonalna.
-11- več ne bo

V primeru ① smo oba lastna vektorja, ki pripadata večkratni lastni vrednosti, izločili iz ene matrike ($C'(\lambda)$).
V primeru ② pa iz dveh matrik ($C(\lambda)$, $C'(\lambda)$). Izločanje lastnih vektorjev iz matrike $C^{(k)}(\lambda)$ vpliva na obliko Jordanove kanonične forme.

A_{3x3}

$\lambda_1 = \lambda_2 = \lambda_3 = 1$

a) $\{(1), (1), (1)\}$
b) $\{(1,1), (1)\}$
c) $\{(1,1,1)\}$

Izraz a) pomeni, da se lastne vred., četudi so enake, obnašajo kot bi bile različne.
V tem primeru bi tri lastne vektorje dobili iz matrike $C(\lambda)$ ALI $C'(\lambda)$ ALI $C''(\lambda)$

Izraz b) pomeni, da se dve lastni vred. zločita skupaj in ena posebej.
V tem primeru bi lastne vektorje dobili iz katerihkoli dveh matrik.
$C(\lambda), C'(\lambda)$ ALI $C(\lambda), C''(\lambda)$ ALI $C'(\lambda), C''(\lambda)$

V c) primeru bomo iz vsake matrike dobili en lastni vektor.
$C(\lambda) \rightarrow u_1$
$C'(\lambda) \rightarrow u_2$
$C''(\lambda) \rightarrow u_3$

a) $J = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$

b) $J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & \\ & & 1 \end{bmatrix}$

c) $J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Jord. kan. f. je najbolj prazna matrika, ki jo lahko dobimo s preslikavo osnovne matrike A.

$J = \Theta^{-1} \cdot A \cdot \Theta$

$e^{At} = \Theta e^{Jt} \cdot \Theta^{-1}$

$A_{7\times 7}: \quad \lambda_1, \underbrace{\lambda_2, \lambda_2, \lambda_2}_{3\times}, \underbrace{\lambda_3, \lambda_3}_{2\times}, \lambda_4$

$$\{(\lambda_2, \lambda_2, \lambda_2)\} \qquad \{(\lambda_3), (\lambda_3)\}$$

$$J = \begin{bmatrix} \lambda_1 & & & & & & \\ & \lambda_2 & 1 & & & 0 & \\ & & \lambda_2 & 1 & & & \\ & & & \lambda_2 & & & \\ & & & & \lambda_3 & & \\ & 0 & & & & \lambda_3 & \\ & & & & & & \lambda_4 \end{bmatrix}$$

$$A \to J = \Theta^{-1} \cdot A \cdot \Theta$$

ALTERNATIVNI POSTOPEK DOLOČANJA JORDANOVE KANONIČNE FORME

A

$g(\lambda) = |A - \lambda I| = 0$

$\lambda_i \ldots$ m-kratna

$[A - \lambda_i I]^k, \quad k = 0, 1, \ldots, m$

$r_k = \text{rang}\{[A - \lambda_i I]^k\}$

$r_0, r_1, r_2, \ldots, r_m$

$r_k = r_{k+1}$

$[(r_0 - r_1), (r_1 - r_2), \ldots, (r_{m-1} - r_m)]$

Segré

Ferrer

$$\begin{array}{ccc|l} * & * & * & r_0 - r_1 \\ * & * & & r_1 - r_2 \\ & \vdots & & \\ \hline \Sigma & \Sigma & \Sigma & \Rightarrow J \end{array}$$

$A_{6\times 6} \to \lambda$

$\lambda: \; 2, \underbrace{1, 1, 1, 1, 1}_{5\times}$

$$J = \begin{bmatrix} 2 & & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & 1 & & \\ & 0 & & & 1 & 1 \\ & & & & & 1 \end{bmatrix}$$

$[A - \lambda_i I]^k, \quad k = 0, 1, \ldots 5$

$[A - I]^k$

$r_0 = 6$

$r_1 = 4$

$r_2 = 2$

$r_3 = 1$

$r_4 = 1$

$[2, 2, 1]$ Segré

$$\begin{array}{cc} * & * \\ * & * \\ * & \\ \hline 3 & 2 \end{array}$$

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$\lambda_1 = 5, \quad \lambda_2 = \lambda_3 = 1$

$[A - \lambda_i I]^k, \quad k = 0, 1, 2$

$[A - I]^k$

$r_0 = 3$

$$[A - I] = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$r_1 = 1$

$[A - I]^2$

$r_2 = 1$

$A \to J \to e^{Jt} \} \to e^{At} = \Theta \cdot e^{Jt} \cdot \Theta^{-1}$

$A \to J \xrightarrow{?} e^{Jt}$

a) $\lambda_i \neq \lambda_j \quad i \neq j$

$$J = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{bmatrix}$$

$$e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \end{bmatrix}$$

b) $(\lambda - \lambda_i)^m$

$$J = \begin{bmatrix} \lambda_1 & 1 & 0 \\ & \lambda_1 & 1 \\ 0 & & \lambda_1 \end{bmatrix}$$

$$J = \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_1 & \\ 0 & & \lambda_1 \end{bmatrix}}_{\lambda_1 I} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{C}$$

e^{Jt}

$Jt = \lambda_1 I \cdot t + Ct$

$e^{Jt} = e^{(\lambda_1 I + C)t} = e^{(\lambda_1 I \cdot t + Ct)} = e^{\lambda_1 I \cdot t} \cdot e^{Ct}$

$$e^{Jt} = e^{\left\{ \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_1 & \\ 0 & & \lambda_1 \end{bmatrix} t + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} t \right\}}$$

$$e^{Jt} = e^{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} t} \cdot e^{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} t}$$

$e^{Ct} = I + Ct + \frac{(Ct)^2}{2!} + \left[\ldots \to 0 \right.$ vrsta

$$C^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$n > 2 \quad C^n = 0$

$$e^{Ct} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & t^2/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e^{Ct} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}, \quad e^{\lambda t} = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix}$$

$$e^{Jt} = e^{Ct} \cdot e^{\lambda t} = \begin{bmatrix} e^t & te^t & \frac{t^2}{2}e^t \\ & e^t & te^t \\ 0 & & e^t \end{bmatrix}$$

Zgleda:

$$J = \begin{bmatrix} \lambda_1 & 1 & & 0 \\ & \lambda_1 & & \\ & & \lambda_1 & \\ 0 & & & \lambda_1 \end{bmatrix}, \quad e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & & 0 \\ & e^{\lambda_1 t} & & \\ & & e^{\lambda_1 t} & \\ 0 & & & e^{\lambda_1 t} \end{bmatrix}$$

$$J = \begin{bmatrix} \lambda_1 & 1 & & 0 \\ & \lambda_1 & 1 & \\ & & \lambda_1 & \\ 0 & & & \lambda_1 \end{bmatrix}, \quad e^{Jt} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \frac{t^2}{2}e^{\lambda_1 t} & 0 \\ & e^{\lambda_1 t} & te^{\lambda_1 t} & 0 \\ & & e^{\lambda_1 t} & 0 \\ 0 & & & e^{\lambda_1 t} \end{bmatrix}$$

$$e^{Jt}$$

$$e^{At} = \Theta \cdot e^{Jt} \cdot \Theta^{-1}$$

a) $a_n \frac{d^n y}{dt^n} + \ldots \quad / \mathcal{L}$

$y(0), y'(0), \ldots y^{(n-1)}(0) = 0$

$H(s) = \frac{Y(s)}{U(s)}$

$\boxed{Y(s) = H(s) \cdot U(s)}$

$H(s) = \frac{P(s)}{Q(s)}$

$Q(s) = 0$

b) $\dot{x} = Ax + Bu$
$y = Cx + Du$ $\Rightarrow H(s) = ?$

Kako poiščemo prenosno funkcijo ?

$$H(s) = \begin{bmatrix} H_{11} & H_{12} & \ldots \\ H_{21} & & \\ \vdots & & \ldots \end{bmatrix}$$

$\dot{x} = Ax + Bu$
$y = Cx + Du \quad / \mathcal{L}$
$x(0) = 0$

prenosna f. ni odvisna od zač. stanj

$s \cdot X(s) = A \cdot X(s) + B \cdot U(s)$
$Y(s) = C \cdot X(s) + D \cdot U(s)$

$sX(s) - A \cdot X(s) = BU(s)$
$(sI - A) \cdot X(s) = B \cdot U(s)$

$$X(s) = (sI - A)^{-1} \cdot B\, U(s)$$
$$Y(s) = C \cdot X(s) + D \cdot U(s)$$
$$Y(s) = C \cdot (sI - A)^{-1} \cdot B \cdot U(s) + D \cdot U(s)$$

$$Y(s) = [\underbrace{C(sI - A)^{-1} \cdot B + D}_{H(s)}] \cdot U(s)$$

$$\boxed{H(s) = C \cdot (sI - A)^{-1} \cdot B + D}$$

Primer:

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \cdot x + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u \quad , \quad y = [6 \; -6 \; 1] \cdot x$$
$$D = 0$$

$$H(s) = C(sI - A)^{-1} \cdot B$$

$$(sI - A) = \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+2 & 0 \\ 0 & 0 & s+3 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{\det} \begin{bmatrix} (s+2)(s+3) & 0 & 0 \\ 0 & (s+1)(s+3) & 0 \\ 0 & 0 & (s+1)(s+2) \end{bmatrix}$$

$$\det = (s+1)(s+2)(s+3)$$

$$(sI - A)^{-1} \cdot B = \frac{1}{\det} \begin{bmatrix} (s+2)(s+3) \\ (s+1)(s+3) \\ (s+1)(s+2) \end{bmatrix}$$

$$C \cdot (sI - A)^{-1} \cdot B = \frac{1}{\det} [6 \cdot (s+2)(s+3) - 6(s+1)(s+3) + (s+1)(s+\cancel{3}^{2})] =$$
$$= \frac{6}{s+1} - 6 \frac{1}{s+2} + \frac{1}{s+3}$$

$$h(t) = \mathcal{L}^{-1}[H(s)] = 6 e^{-t} - 6 e^{-2t} + e^{-3t}$$

POVZETEK 3. POGLAVJA

$$\dot{x} = Ax + Bu$$

$$x(t) = e^{At} \cdot x(0) + \int_0^t e^{A(t-\tau)} \cdot B \cdot u(\tau)\, d\tau$$

$$y(t) = C \cdot x(t) + D \cdot u(t)$$

$$e^{At}$$

① $e^{At} \approx I + At + \frac{(At)^2}{2!} + \ldots$

② $e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$

③ $e^{At} = \sum_{k=0}^{n-1} \alpha_k A^k$

$$e^{\lambda t} = \sum_{k=0}^{n-1} \alpha_k \lambda^k \Rightarrow \alpha_k$$

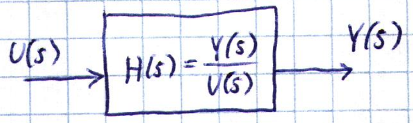
$$A \to J \to e^{Jt} \to e^{At}$$
$$e^{At} = Q \cdot e^{Jt} \cdot Q^{-1}$$

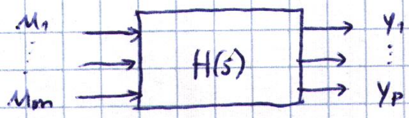
$$Q = [u_1 \vdots u_2 \vdots \ldots]$$

$$H(s) = C(sI - A)^{-1} \cdot B + D$$

IV. MODELIRANJE SISTEMOV Z BLOČNIMI DIAGRAMI

V bločnem diagramu so vse spremenljivke med seboj povezane s funkcionalnimi bloki, ki ponazarjajo zvezo med vhodom in izhodom, zato vsak blok predstavlja prenosno funkcijo.



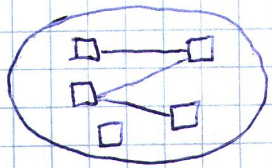


$$H(s) = \begin{bmatrix} H_{11} & H_{12} & \dots & H_{1m} \\ & & & \\ H_{p1} & \dots & & H_{pm} \end{bmatrix}$$

preslika vse vhode v vse izhode

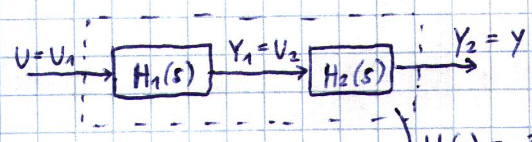
$$Y(s) = H(s) \cdot U(s)$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = H(s) \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$



IV. 1. ALGEBRA BLOČNIH DIAGRAMOV

① Zaporedna



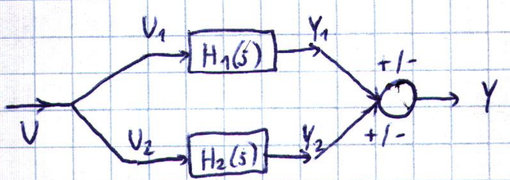
$$H(s) = \frac{Y(s)}{U(s)}$$

$$\underline{Y(s) = H(s) \cdot U(s)}$$

$$Y = Y_2 = H_2 \cdot U_2 = H_2 \cdot Y_1 = H_2 H_1 U_1 = \underline{\underline{H_2 \cdot H_1 \cdot U}}$$

$$\boxed{H(s) = H_2(s) \cdot H_1(s)}$$

② Vzporedna

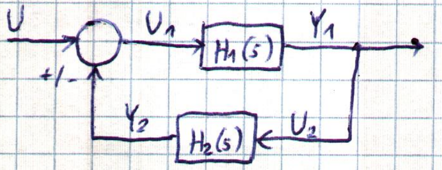


$$Y = \pm Y_1 \pm Y_2 = \pm H_1 \cdot U_1 \pm H_2 \cdot U_2 =$$

$$= \pm H_1 \cdot U \pm H_2 \cdot U = (\pm H_1 \pm H_2)\, U$$

$$\boxed{H(s) = \pm H_1(s) \pm H_2(s)}$$

③ Povratna zanka



$$Y = Y_1 = H_1 \cdot U_1 = H_1 (U \pm Y_2) = H_1 \cdot (U \pm H_2 U_2) =$$

$$= H_1 (U \pm H_2 Y) = H_1 U \pm H_1 H_2 Y$$

$$Y \mp H_1 H_2 Y = H_1 U$$

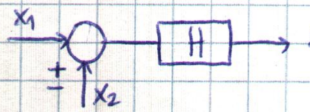
$$(I \mp H_1 H_2) Y = H_1 U$$

$$Y = (I \mp H_1 H_2)^{-1} \cdot H_1 \cdot U$$

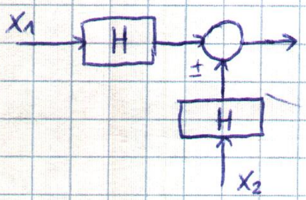
$$\boxed{H(s) = (I \mp H_1 H_2)^{-1} \cdot H_1}$$

$$H(s) = \frac{H_1}{1 \mp H_1 H_2}$$

Premik seštevalnika za blok



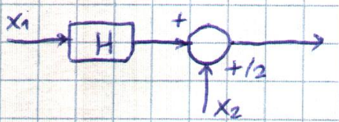
$x_3 = H(x_1 \pm x_2)$



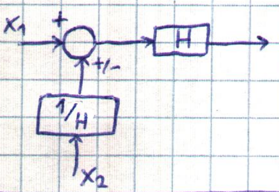
$x_3 = H(x_1 \pm x_2)$

Kaj moramo pripeljat, da dobimo x_3?

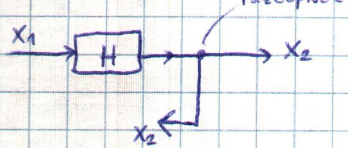
Premik seštevalnika pred blok



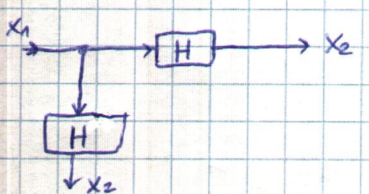
$x_3 = H \cdot x_1 \pm x_2$

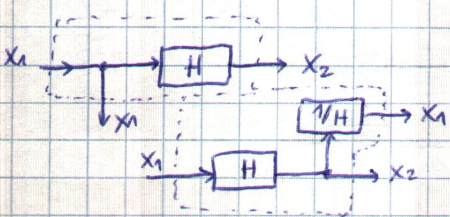


$x_3 = H \cdot x_1 \pm x_2$



razcepišče pred blok

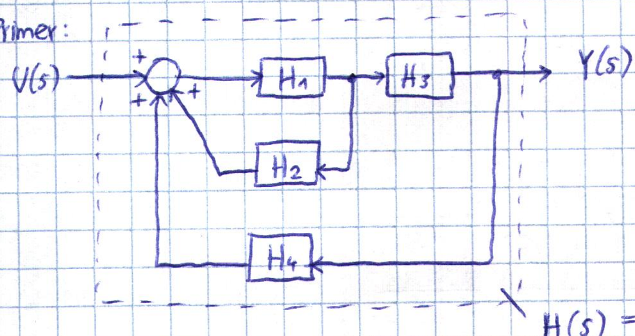




REDUKCIJA BLOKOVNEGA DIAGRAMA

Kako poiskat celotno funkcijo vseh podsistemov ?

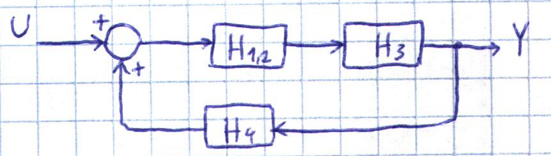
Primer:



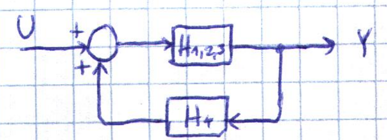
$H(s) = ?$

$Y(s) = H(s) \cdot U(s)$

$$H_{1,2} = \frac{H_1}{1 - H_1 \cdot H_2}$$



$$H_{1,2,3} = H_3 \cdot H_{1,2}$$



$$H = H_{1,2,3,4} = \frac{H_{1,2,3}}{1 - H_{1,2,3} \cdot H_4} = \frac{H_3 \cdot H_{1,2}}{1 - H_3 \cdot H_{1,2} \cdot H_4} =$$

$$= \frac{H_3 \cdot \frac{H_1}{1 - H_1 \cdot H_2}}{1 - H_3 \cdot \frac{H_1}{1 - H_1 H_2} \cdot H_4}$$

Kako iz bločnega diagrama dobimo enačbe stanj ?

$\dot{x} = Ax + Bu$

$y = Cx + Du$

$$\begin{cases} \dot{x} = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot u \\ y = [1 \;\; 0] \cdot x \end{cases}$$

$\dot{x}_1 = -3x_1 - x_2 + u$

$\dot{x}_2 = 2x_1$

$y = x_1$

$/ \; \mathcal{L}$

$sX_1(s) = -3 \cdot X_1(s) - X_2(s) + U(s)$

$sX_2(s) = 2X_1(s)$

$Y(s) = X_1(s)$

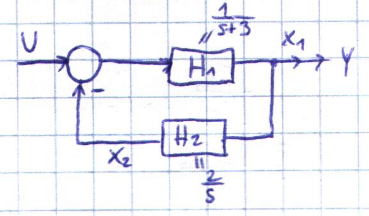
$$\begin{cases} (s+3)X_1(s) = -X_2(s) + U(s) \\ sX_2(s) = 2X_1(s) \\ Y(s) = X_1(s) \end{cases}$$

$Y(s) = H(s) \cdot U(s)$

$X_1(s) = -\frac{1}{s+3} X_2 + \frac{1}{s+3} U = \frac{1}{s+3}(-X_2 + U)$

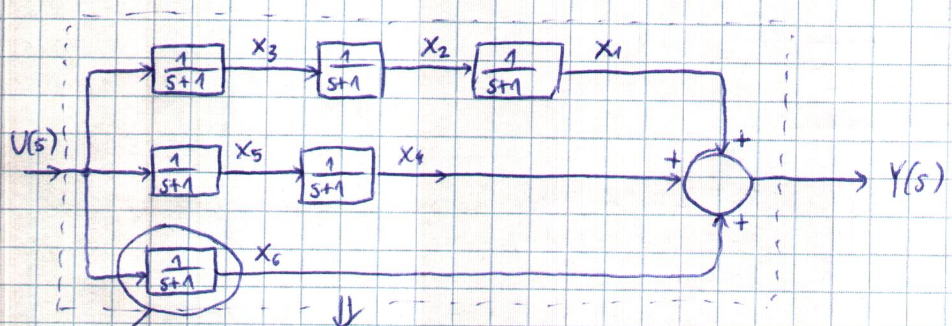
$X_2(s) = \frac{2}{s} X_1(s)$

$Y(s) = X_1(s)$



$H_1(s) = \frac{1}{s+3}$

$H_2(s) = \frac{2}{s}$



$$\Downarrow$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
?

$Y = H \cdot U$

$\dot{x} = Ax + Bu \quad / \mathcal{L}$

$sX = AX + BU$

$X_1(s) = \frac{1}{s+1} X_2(s)$

$(s+1) X_1 = X_2$

$sX_1 + X_1 = X_2$

$sX_1 = -X_1 + X_2$

$\dot{x}_1 \quad \mathcal{L}^{-1}$

$\dot{x}_1 = -x_1 + x_2$
$\dot{x}_2 = -x_2 + x_3$
$\dot{x}_3 = -x_3 + u$
$\dot{x}_4 = -x_4 + x_5$
$\dot{x}_5 = -x_5 + u$
$\dot{x}_6 = -x_6 + u$

$X_3 = \frac{1}{s+1} U$

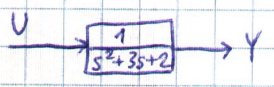
$(s+1) X_3 = U$

$sX_3 + X_3 = U \quad / \mathcal{L}^{-1}$

$\dot{x}_3 = -x_3 + u$

$Y = X_1 + X_4 + X_6$

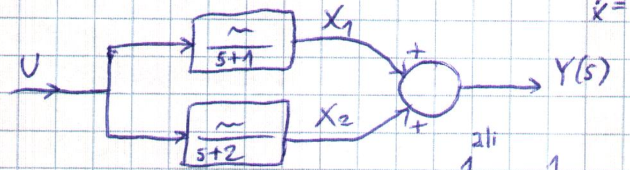
$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} u \quad , \quad y = [1\,0\,0\,1\,0\,1] \cdot x$$



$$\frac{1}{s^2+3s+2} = \frac{\sim}{s+1} + \frac{\sim}{s+2}$$

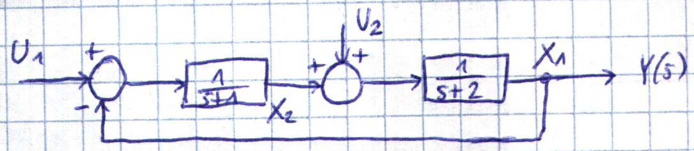
$s^2 + 3s + 2 = (s+1)(s+2)$

$\dot{x} = Ax + Bu$



ali

$$\frac{1}{s+1} \cdot \frac{1}{s+2}$$



$$X_1 = \frac{1}{s+2}(X_2 + U_2)$$
$$(s+2)X_1 = X_2 + U_2$$
$$\underline{sX_1 = -2X_1 + X_2 + U_2 \quad / \mathcal{L}^{-1}}$$
$$X_2 = \frac{1}{s+1}(U - X_1)$$

$$(s+1)X_2 = U - X_1$$
$$\underline{sX_2 = -X_1 - X_2 + U \quad / \mathcal{L}^{-1}}$$

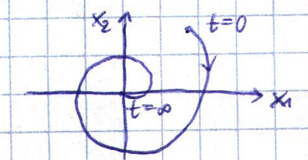
$$\underline{Y = X_1 \quad / \mathcal{L}^{-1}}$$

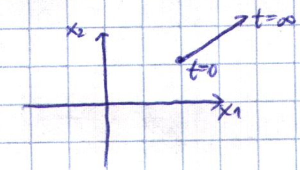
$$\dot{x}_1 = -2x_1 + x_2 + u_2$$
$$\dot{x}_2 = -x_1 - x_2 + u_1$$
$$\underline{y = x_1}$$

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

V. RAVNOTEŽNA STANJA SISTEMA

$$\dot{x} = Ax + Bu$$
$$\Downarrow$$
$$x = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \end{bmatrix}$$





Med stanji, kijih lahko nek sistem zavzame, so še posebej pomembna ravnotežna stanja. Ravnotežno stanje je stanje, ki se ne spreminja, ko sistem prepustimo samemu sebi.

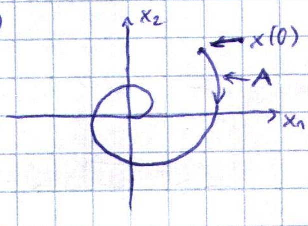
$$x(t) = \overbrace{e^{At} \cdot x(0)} + \overbrace{\int_0^t e^{A(t-\tau)} \cdot B \cdot u(\tau)\, d\tau}$$

$A, B, x(0), u(t)$

$u(t) = 0$

$A, x(0)$

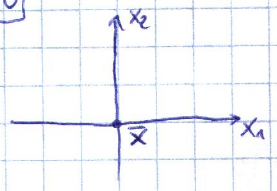
$\dot{x} = Ax$



če je $u(t) = 0$, vpliva le osn. matrika A

$$u(t) = 0$$
$$\dot{x} = Ax$$
$$\dot{x} = 0$$
$$A \cdot \bar{x} = 0$$
$$\boxed{\bar{x} = 0}$$

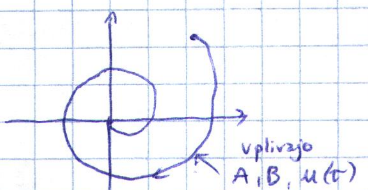
$\bar{x} \rightarrow$ smo v ravnot. stanju



kijih ne vzbujamo

Linearni sistemi imajo eno samo ravnotežno stanje, ki se nahaja v koord. izhodišču. prostora stanj.

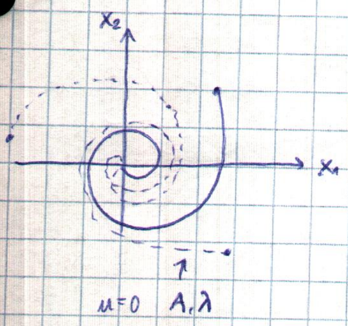
$u(t) \neq 0, \quad x(0) \neq 0$



$$\dot{x} = Ax + Bu$$
$$\dot{x} = 0$$
$$A\bar{x} + Bu = 0$$
$$A\bar{x} = -Bu \quad / A^{-1}$$
$$\boxed{\bar{x} = -A^{-1} \cdot B \cdot u}$$

FAZNI PORTRET

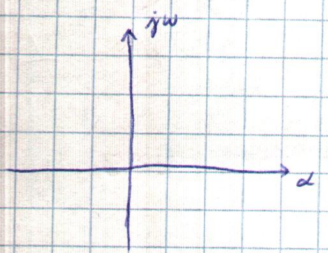
Faz. portret je množica tirnic, ki so izrisane v okolici ravnotežnega stanja.



štartamo iz razl. točk

Ravnotežna stanja linearnih sistemov se med seboj razlikujejo po faznih portretih.

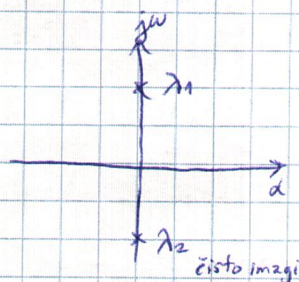
$$A_{2x2} \Rightarrow \lambda_1, \lambda_2$$

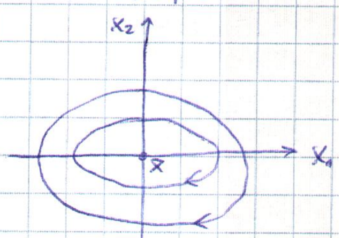


$$e^{\lambda_i t} \qquad e^{At} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & \ddots \end{bmatrix}$$

① 1. tip **Središče**

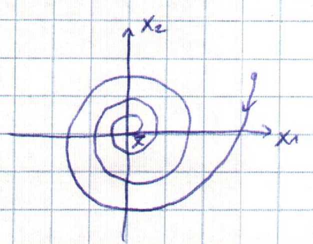
Ravnot. stanje središče ponazarja točko v prostoru stanj, ki jo tirnice popolnoma obkrožajo.
↳ elipse

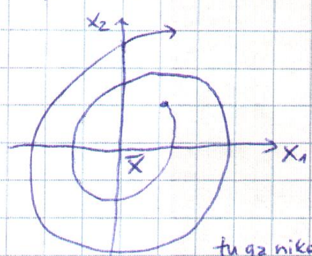




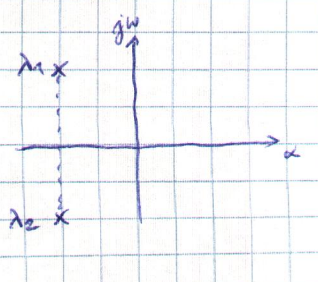
② **Žarišče**

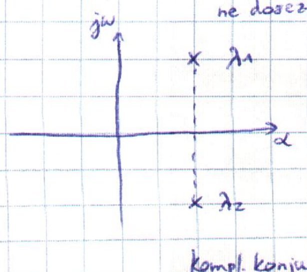
Ravnot. stanje žarišče ponazarja točko v prostoru stanj, ki je ali ponor ali izvor tirnic v obliki spiral.





tu ga nikoli ne dosežem

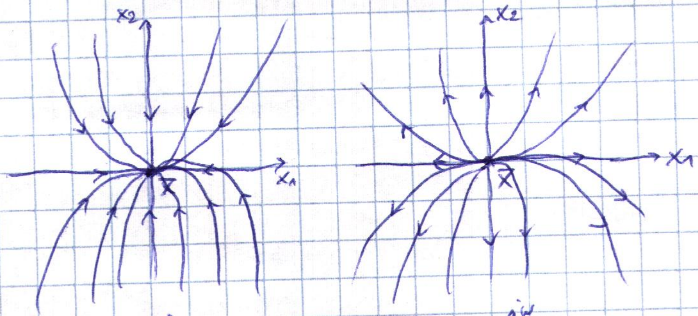


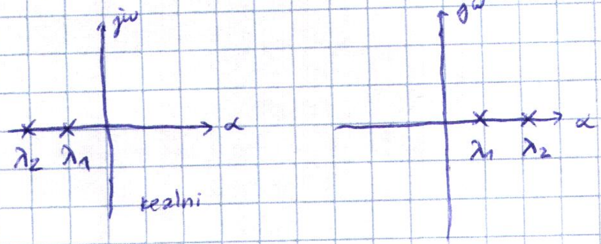


kompl. konjug. pari

③ Vozlišče

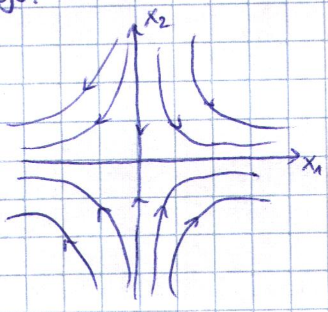
Ravnot. stanje vozlišče ponazarja točko v prostoru stanj, ki je ali ponor ali izvor tirnic poljubne oblike.

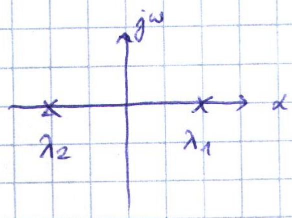




④ Sedlo

Ravnot. stanje sedlo ponazarja točka v prostoru stanj, kateri se tirnice najprej približajo, potem pa se od nje oddaljujejo.





Primer: $\dot{x}_1 = 2x_1 - x_2$

$\dot{x}_2 = x_1 - 3x_2$

$$\dot{x} = \begin{bmatrix} 2 & -1 \\ 1 & -3 \end{bmatrix} x$$

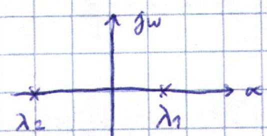
$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} = 0$$

$\lambda^2 + \lambda - 6 + 1 = 0$

$\lambda^2 + \lambda - 5 = 0$

$\lambda_{1,2} = ?$

$$\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{21}}{2}$$



sedlo

$\dot{x}_1 = x_1 - 2x_2$

$\dot{x}_2 = 4x_1 - x_1^3$ nelin. sistem, ker imamo x_1^3

$\dot{x}_1 = \dot{x}_2 = 0$

nelin. sistemi imajo v splošnem več kot eno ravnot. stanje

$\bar{x}_1 - 2\bar{x}_2 = 0$

$4\bar{x}_1 - \bar{x}_1^3 = 0 \quad \longrightarrow \quad \bar{x}_1 \underbrace{(4 - \bar{x}_1^2)} = 0$

$\bar{x}_1 = 0$

$\bar{x}_2 = 0$

$4 - \bar{x}_1 = 0$

$\bar{x}_1 = 2 \qquad \bar{x}_1 = -2$

$\bar{x}_2 = 1 \qquad \bar{x}_2 = -1$

ZA VAJO:

$\dot{x}_1 = x_1 - 2x_2$

$\dot{x}_2 = 4x_1 - x_2^3$

Ravnotežno stanje je asimptotično stabilno, če so realni deli lastnih vrednosti negativni, stabilno (vendar ne-asimptotično stabilno), če je lastna vred. na imag. osi (realna vred. je 0) in nestabilno, če je realna vrednost katerekoli lastne vred. pozitivna.

VI. VODLJIVOST in SPOZNAVNOST SISTEMOV

$$u(t) \longrightarrow \boxed{h(t)\quad S} \longrightarrow y(t)$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Način ali izraz

$$g(\lambda) = |A - \lambda I| = 0$$
$$\Downarrow$$
$$\left\{ \begin{matrix} \lambda_1, \lambda_2, \dots \lambda_n \\ e^{\lambda_1 t}, e^{\lambda_2 t}, \dots e^{\lambda_n t} \end{matrix} \right\}$$

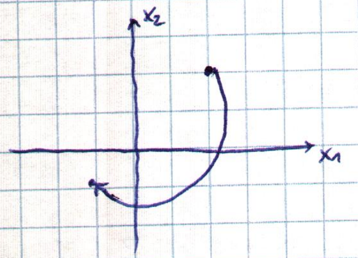
To množico imenujemo spekter izrazov sistema.

n-lastnim vrednostim pripada n izrazov sistema.

$$\searrow \quad e^{\lambda_i t}, \quad i = 1, 2, \dots n$$

Vodljivost

Zvezni sistem je vodljiv, če ga je mogoče s poljubnim vhodnim signalom $u(t)$ v končnem času privesti iz kateregakoli začetnega stanja v katerokoli končno stanje.



Da bi bil sistem vodljiv, morajo biti vsi njegovi načini dostopni (speti) z vhodnim signalom. Če je ta pogoj izpolnjen, lahko poljubno spreminjamo stanje sistema, tako da v določenem času spreminjamo vhodni signal oz. vzbujanje.

$$x(t) = e^{At} \cdot x(0) + \int_0^t e^{A(t-\tau)} \cdot B u(\tau)\, d\tau$$

Kako testiramo, ali je sistem vodljiv ali ne?

tvorimo matriko $M = [\, B \;\vdots\; AB \;\vdots\; A^2B \;\vdots\; \dots \,]$

Sistem je vodljiv, če in samo če so stolpci matrike M linearno neodvisni.

$$\leadsto \boxed{\begin{matrix} \text{rang } M = n \dots \text{vodljiv} \\ \qquad < n \dots \text{ni vodljiv} \end{matrix}}$$

$A_{n \times n}$

zgled: $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$M = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{rang } M < 3 \longrightarrow \text{Sistem ni vodljiv.}$$

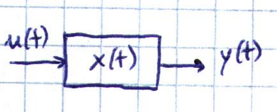
$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix} \quad \rightarrow \text{rang}\, M = 3$$

Spoznavnost

Zvezni sistem je spoznaven, če je mogoče določiti njegovo stanje $x(t)$ na osnovi končno dolgega opazovanja izhodnega signala $y(t)$.



Test spoznavnosti

$$N = [\, C^T \,\vdots\, A^T . C^T \,\vdots\, (A^T)^2 C^T \,\vdots\, \ldots \,]$$

$A_{n \times n}$ $\quad N_{n \times n}$

Potreben in zadosten pogoj, da je sistem spoznaven, je, da je rang N enak n.

$$\boxed{\begin{aligned} \text{rang}\, N &= n \;\ldots\; \text{spoznaven} \\ &< n \;\ldots\; \text{ni spoznaven} \end{aligned}}$$

Zgled:

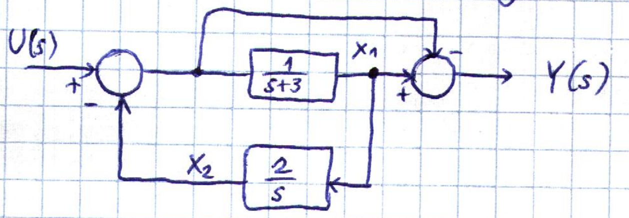
$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad A^T = A, \quad C = [1 \; 0 \; 2], \quad C^T = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 2 & -6 & 18 \end{bmatrix} \quad \rightarrow \text{rang}\, N < 3 \quad \rightarrow \text{Sistem ni spoznaven.}$$

Če je sistem opisan z bločnim diagramom ...

Je sistem vodljiv, spoznaven ?

M ... A, B
N ... A, C



$$X_1 = \frac{1}{s+3}(-X_2 + U)$$
$$(s+3)X_1 = -X_2 + U$$
$$sX_1 = -3X_1 - X_2 + U \quad / \mathcal{L}^{-1}$$
$$\dot{x}_1 = -3x_1 - x_2 + u$$

kar je na vhodu

$$X_2 = \frac{2}{s} X_1$$
$$sX_2 = 2X_1 \quad / \mathcal{L}^{-1}$$
$$\dot{x}_2 = 2x_1$$

$$\dot{x} = \underbrace{\begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u$$

$$Y = X_1 - (-X_2 + U)$$
$$Y = X_1 + X_2 - U$$
$$y = \underbrace{[1 \;\; 1]}_{C} \cdot x - u$$

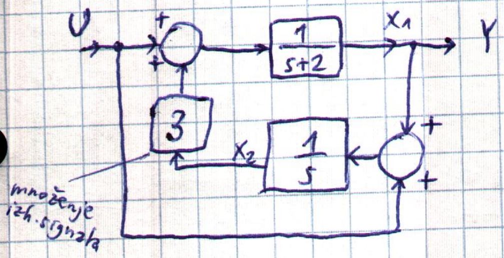
$$M = [\, B \,\vdots\, A.B \,]$$

Sistem je vodljiv.

$$M = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} \rightarrow \text{rang}\, M = 2$$

$$C^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$$

$$N = [\, C^T \,\vdots\, A^T . C^T \,] = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \rightarrow \text{rang}\, N = 1$$

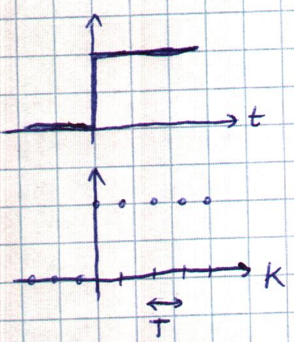


DN

VII. DISKRETNI SISTEMI

$u(t) \rightarrow u[k]$
$x(t) \rightarrow x[k]$
$y(t) \rightarrow y[k]$

zvezni diskretni



$u[kT] \ldots u[k]$

kT je diskretna časovna spremenljivka
T je fiksna pozitivna konstanta

① $y[k+n] + a_{n-1} \cdot y[k+n-1] + a_{n-2} y[k+n-2] + \ldots + a_1 y[k+1] + a_0 y[k] =$
$= b_m u[k+m] + \ldots + b_1 u[k+1] + b_0 u[k]$

Ta zapis je ekvivalenten zapisu z odvodi pri zveznih sistemih.

$y[k] = ?$, $k = 0, 1, 2, \ldots$

$y[0], y[1], \ldots y[n-1]$ → začetni pogoji

$y[k]$ $k \geq n$

Ta zapis ① je v skladu s splošno matematično teorijo linearnih diferenčnih enačb s konst. koeficienti. Takšen zapis dobimo tudi po diskretizaciji lin. diferencialnih enačb s konst. koef. in ta zapis srečamo tudi pri numeričnem reševanju diferencialnih enačb.

② $y[k]+a_{n-1}\,y[k-1]+\ldots+a_1\,y[k-(n-1)]+a_0\,y[k-n]=$

$= b_m\cdot u[k-(n-m)]+\ldots+b_1\,u[k-(n-1)]+b_0\,u[k-n]$

$y[k]=?\quad,\quad k=0,1,2,\ldots$

$y[-1],\ y[-2],\ \ldots,\ y[-n]$

Ta zapis se uporablja pri lin. digitalnih filtrih.

① $\longleftrightarrow$ ②

diferenč. enačba 2. reda

① $y[k+2]-3y[k+1]+2\,y[k]=u[k+1]$

$y[0]=0\ ,\ y[1]=-4$

$u[k]=3^k\cdot u^*[k]$

↖ diskretna enotina stopnica

② $y[k]-3y[k-1]+2y[k-2]=u[k-1]$

$y[-1],\ y[-2]$

$k=-1$

$\underset{\checkmark}{y[1]}-\underbrace{3\underset{\checkmark}{y[0]}}_{0}+2\underset{?}{y[-1]}=u[0]$

$-4+2y[-1]=1$

$y[-1]=\frac{5}{2}$

(to je en zač. pogoj)

$k=-2$

$\underbrace{\underset{\checkmark}{y[0]}}_{0}-3\underset{\checkmark}{y[-1]}+2\underset{?}{y[-2]}=u[\cancel{-1}]$

$y[-2]=\frac{15}{4}$

$\mathcal{L}\qquad\qquad \mathcal{Z}$

$f(t)\ /\mathcal{L}\ =\ F(s)$

$$\boxed{F(s)=\int_0^\infty f(t)\cdot e^{-st}\,dt}$$

$f[k]\ /\mathcal{Z}\ =\ F(z)$

$$\boxed{F[z]=\sum_{k=0}^{\infty} f[k]\cdot z^{-k}}$$

Lastnosti Z preslikave:

1. Linearnost

$$\mathcal{Z}[\alpha_1 \cdot f_1[k] + \alpha_2 \cdot f_2[k]] = \alpha_1 F_1[z] + \alpha_2 F_2[z]$$ vsota Z preslikav

2. Premik v desno ("integriranje")

$$\mathcal{Z}[f[k] \cdot u[k]] = F[z]$$

$$\mathcal{Z}[f[k-k_0] \cdot u[k-k_0]] = \frac{1}{z^{k_0}} \cdot F[z]$$

3. Premik v levo ("odvajanje")

$$\mathcal{Z}[f[k] \cdot u[k]] = F[z]$$

$$\mathcal{Z}[f[k+1] \cdot u[k]] = z \cdot F[z] - z \cdot f[0]$$

$$\mathcal{Z}[f[k+2] \cdot u[k]] = z^2 \cdot F[z] - z^2 \cdot f[0] - z f[1]$$

$$\vdots$$

$$\mathcal{Z}[f[k+k_0] \cdot u[k]] = z^{k_0} F[z] - z^{k_0} [f(0)] - z^{(k_0-1)} \cdot f[1] - z \cdot f[k_0-1]$$

4. $\mathcal{Z}[f[k]] = F[z]$

$$\mathcal{Z}[a^k \cdot f[k]] = F\left[\frac{z}{a}\right]$$

$$\mathcal{Z}[a^{-k} \cdot f[k]] = F[a \cdot z]$$

5. Diskretna konvolucija

$$f_1[k] * f_2[k] = \sum_{n=0}^{\infty} f_1[m] \cdot f_2[k-m]$$

$$\mathcal{Z}[f_1[k] * f_2[k]] = F_1[z] \cdot F_2[z]$$

$f[k]$	$F[z]$
$\delta[k] = \begin{cases} 1 & k=0 \\ 0 & \text{drugje} \end{cases}$	1
$u[k]$	$\frac{z}{z-1}$
a^k	$z/(z-a)$
k	$z/(z-1)^2$
k^2	$\frac{z(z+1)}{(z-1)^2}$
$k \cdot a^k$	$\frac{z \cdot k}{(z-a)^2}$
$\begin{cases} a^{k-1} & k \geq 1 \\ 0 & \end{cases}$	$\frac{1}{z-a}$

$$y[k+n] + a_{n-1}\, y[k+n-1] + \ldots + a_1\, y[k+1] + a_0\, y[k] = u[k] \quad / \mathcal{Z}$$

$$y[0],\ y[1], \ldots,\ y[n-1]$$

$$z^n \cdot Y[z] - z^n \cdot y[0] - \ldots - z \cdot y[n-1]] + a_{n-1}[z^{n-1} \cdot Y[z] - \ldots] + \ldots +$$

$$+ a_1[zY[z] - z \cdot y[0]] + a_0 \cdot Y[z] = U[z]$$

$$\underbrace{[z^n + a_{n-1} \cdot z^{n-1} + \ldots + a_1 z + a_0]}_{Q[z]}\, Y[z] - \underbrace{[\text{zač. stanja}]}_{P[z]} = U[z]$$

$$Q[z] \cdot Y[z] = P[z] + U[z]$$

$$\boxed{Y[z] = \frac{P[z] + U[z]}{Q[z]}}$$

$Q[z] = 0$

$\rightarrow$ dobimo značilno enačbo

$$Q[z] = 0 \Rightarrow z_1, z_2, \ldots, z_n$$

$$Y[z] = \frac{P[z] \cdot U[z]}{(z-z_1)(z-z_2)\ldots(z-z_n)}$$

$$Y[z] = \frac{A_1 \cdot z}{(z-z_1)} + \frac{A_2 \cdot z}{(z-z_2)} + \ldots + \frac{A_n\, z}{(z-z_n)}$$

$$\mathcal{Z}^{-1}\left[\frac{A_i \cdot z}{(z-z_i)}\right] = A_i \cdot z_i^k$$

$$y[k] = A_1 \cdot z_1^k + A_2 \cdot z_2^k + \ldots$$

$$A_1 e^{s_1 t} + A_2 \cdot e^{s_2 t} + \ldots \quad \text{(zvezni)}$$

DN: ① $x[k+2] = x[k+1] + x[k]$

$x[0] = 0,\ x[1] = 1,\quad x[k] = ?$

② $x[z] = \dfrac{1}{(1-\frac{1}{z})(1+\frac{0.5}{z})}$

$x[k] = ?$

$$y[k+2] + a_1 \cdot y[k+1] + a_0\, y[k] = b_0 \cdot u[k]$$

$x_1[k],\ x_2[k]$

izberemo: $x_1[k] = y[k]$

1. diferenca: $x_2[k] = x_1[k+1] = y[k+1]$

$x_2[k+1] = x_1[k+2] = y[k+2]$

$x_2[k+1] = -a_1 x_2[k] - a_0 x_1[k] + b_0\, u[k]$

$$\begin{cases} x_1[k+1] = x_2[k] \\ x_2[k+1] = -a_0 x_1[k] - a_1 x_2[k] + b_0\, u[k] \end{cases}$$

$$x[k+1] = \begin{bmatrix} x_1[k+1] \\ x_2[k+1] \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}}^{A} \cdot \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ b_0 \end{bmatrix}}^{B} \cdot u[k]$$

$$y[k] = \underbrace{[1 \quad 0]}_{C} \cdot \begin{bmatrix} x_1[k] \\ x_2[k] \end{bmatrix}$$

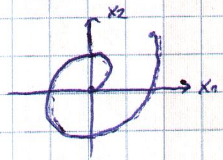
Diskr. sistem modeliramo na ta način.

$$\boxed{\begin{aligned} x[k+1] &= A \cdot x[k] + B \cdot u[k] \\ y[k] &= C \cdot x[k] + D \cdot u[k] \end{aligned}}$$

$x[k] = ?$

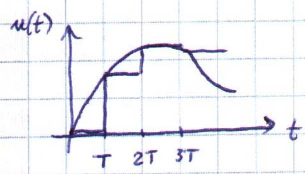
$$x[k+1] = A_d \cdot x[k] + B_d \cdot u[k]$$
$$y[k] = C_d \cdot x[k] + D_d \cdot u[k]$$

$$? \begin{cases} A, B, C, D \Rightarrow x(t) \\ A_d, B_d, C_d, D_d \Rightarrow x[k] \end{cases}$$



Diskretizacija časovno zveznih sistemov

① Integralna aproksimacijska metoda



$$\dot{x} = Ax + Bu$$
$$x(t) = e^{At} \cdot x(0) + \int_0^t e^{A(t-\tau)} \cdot B \cdot u(\tau)\, d\tau$$

Predpostavimo, da je vzbujanje med 2 čas. trenutkoma konstantno.

$t:\ 0 - T \qquad u(t) = u(0)$

$$\begin{cases} x(T) = e^{AT} \cdot x(0) + \int_0^T e^{A(T-\tau)} \cdot B \cdot u(0)\, d\tau \\ x(T) = e^{AT} \cdot x(0) + \left(\int_0^T e^{A(T-\tau)} \cdot B\, d\tau\right) \cdot u(0) \end{cases}$$

$$\begin{cases} x[T(k+1)] = A_d \cdot x[kT] + B_d \cdot u[kT] \\ k = 0 \\ \qquad x[T] = A_d \cdot x[0] + B_d \cdot u[0] \end{cases}$$

$$A_d = e^{AT}$$
$$B_d = \int_0^T e^{A(T-\tau)} \cdot B\, d\tau$$
$$C_d = C$$
$$D_d = D$$

$B_d = (A_d - I) \cdot A^{-1} \cdot B$

$A_d = e^{AT}$

$C_d = C$

$D_d = D$