MAT. VAJE

melita.hajdinjak@fe.uni-lj.si → ime in priimek
ime predmeta.
ime profesorja
vpisna številka
smer (UN)

gov. ure: čet. ob 13h v $\boxed{A307}$

http://matematika.fe.uni-lj.si

repetitorij: četrtek 18-20 v P3.

literatura: teorija: -Tomšič, Mramor-Kosta, Orel: Matematika I.
-Jurčič-Žlobec, Mramor-Kosta: Zbirka nalog iz MAT I.
-Dolinar, Demšar: Rešene naloge iz MAT I. za VSP

POGOJI

- pisni del: primeri starih izpitov na spl. strani: Gradiva
  - pisni izpit, ≥ 50% prijava preko e-studenta MAT 1
  - kolokviji
    1. kolokvij - 24. november
       pogoj: 5/10 DN
       pogoj: obvezna pravočasna prijava preko e-studenta
    2. kolokvij - 14. januar
       pogoj: 30% na prvem kolokviju
       pogoj: 5/10 DN
       povprečje vsaj 50%, da opraviš pisni izpit
- ustni del:
  - ustni izpit, ko opraviš pisni del
    po pisnem izpitu, rok ≤ 3 dni
    - vprašanja so na spl. strani (predmeti, MAT I, vprašanja)
    - s kolokviji: obvezna prijava preko e-študenta
      prijava na PISNI IZPIT, obkljukaš možnost p. del
      opravil s kolokviji

MNOŽICE

$A, B, \ldots$ množice, $x \in A$, $x \notin A$

$\emptyset = \{\}$ ... prazna množica

$\mathcal{U}$ ... univerzalna mn.

$A \subseteq B$ ... podmnožica

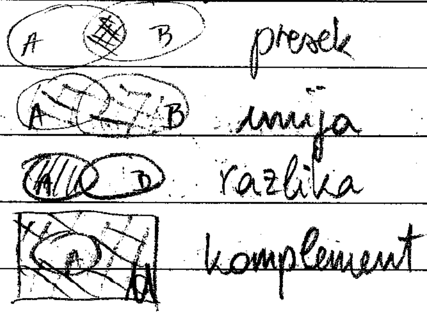
VENOVI DIAGRAMI:

$A \cap B = \{x;\ x \in A \overset{\text{in}}{\wedge} x \in B\}$

$A \cup B = \{x;\ x \in A \overset{\text{ali}}{\vee} x \in B\}$

$A \setminus B = \{x;\ x \in A \wedge x \notin B\}$

$A^c = \{x;\ x \in \mathcal{U} \wedge x \notin A\}$



$A \setminus B = A - B = A \cap B^c$

$A^c = \overline{A}$

$|A| = m(A)$ ... št. elementov v A

$\mathbb{N}$ - $\overset{\text{jih je}}{\text{števno}}$ neskončno

$\mathbb{R}$ - $\overset{\text{jih je}}{\text{neštevno}}$ neskončno

LASTNOSTI:

$A \dot{\cap} (B \overset{+}{\cup} C) = (A \cap B) \cup (A \cap C)$

$A \dot{\cup} (B \overset{+}{\cap} C) = (A \cup B) \cap (A \cup B)$

} DISTRIBUTIVNOST

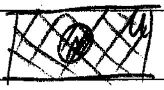
$(A \cap B)^c = A^c \cup B^c$

$(A \cup B)^c = A^c \cap B^c$

} De Morganova zakona

$A \cap A^c = \emptyset$

$A \cup A^c = \mathcal{U} \leftarrow$



$A \cap \emptyset = \emptyset$

$A \cup \emptyset = A$

$A \cap \mathcal{U} = A$

$A \cup \mathcal{U} = \mathcal{U}$

$(A^c)^c = A$

① Naj bo $\mathcal{U} = \mathbb{R}$ univ. množica

f, g naj preslikata v R v preslikavi

A je množica iz vseh x iz $\mathbb{R}$ za katere velja $f(x) = 0$ in

B je mn. vseh x iz $\mathbb{R}$ -11- $g(x) = 0$

Izrazi z množicama A in B rešitve naslednjih enačb

$f, g\ \mathbb{R} \to \mathbb{R}$

$A = \{x \in \mathbb{R};\ f(x) = 0\}$

$B = \{x \in \mathbb{R};\ g(x) = 0\}$

$\boxed{\text{ali} = \vee = \cup}$
$\boxed{\text{in} = \wedge = \cap}$

a) $f(x) \cdot g(x) = 0$

$f(x) = 0$ ali $g(x) = 0$ $\qquad C = \{x \in \mathbb{R}, f(x) \cdot g(x) = 0\} = A \cup B$

! $\boxed{\text{b)}^{\text{DN}}\ f^2(x) + g^2(x) = 0\ ;\ R = A \cap B}$

c) $g^2(x) \cdot f^2(x) + g^2(x) = 0$

$g^2(x)\,(f^2(x) + 1) = 0$ $\qquad D = \{x \in \mathbb{R}, g^2(x) \cdot f^2(x) + g^2(x) = 0\} =$

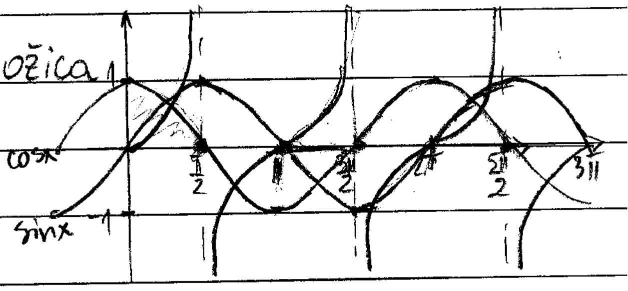
$g^2(x) = 0$ ali $f^2(x) + 1 = 0$ $\qquad = \{x \in \mathbb{R}, g(x) = 0\} = B$

$g(x) = 0$

$\to f^2(x) + g^2(x) = 0\ ,\ f^2(x) = -g^2(x) \Rightarrow f^2(x) = g^2(x)$

$C = \{x \in \mathbb{R};\ f^2(x) + g^2(x) = 0\} = \{x \in \mathbb{R};\ f^2(x) = g^2(x)\} = \underline{\underline{A \cap B}}$

② Naj bo $\mathcal{U} = [0, 3\pi]$ univ. množica



$A = \{x;\ \sin x < 0\}$

$B = \{x;\ \cos x < 0\}$

$C = \{x;\ \tan x \geq 0\}$

Zapiši kot intervale ali unije intervalov naslednje množice

$A, B, C, A \cap B, A^c \cap C, \boxed{B^c \cup C}$! R: $[0, \frac{\pi}{2}] \cup [2\pi, \frac{5\pi}{2}]$! DN

$A = \{x;\ \sin x < 0\} = (\pi, 2\pi)\ ;\ B = \{x; \cos x < 0\} = (\frac{\pi}{2}, \frac{3\pi}{2}) \cup (\frac{5\pi}{2}, 3\pi)$

$C = \{x;\ \tan x > 0\} = (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2}) \cup (2\pi, \frac{5\pi}{2})$

$A \cap B = \{x;\ \sin x < 0 \wedge \cos x < 0\} = (\pi, \frac{3\pi}{2})$

$A^c \cap C = \{x;\ \sin x \geq 0 \wedge \tan x > 0\} = (0, \frac{\pi}{2}) \cup (2\pi, \frac{5\pi}{2})$

PASCALOV TRIKOTNIK

1

$\rightarrow$ 1 1

$a^2 \rightarrow$ 1 2 1

$a^3 \rightarrow$ 1 3 3 1

$a^4 \rightarrow$ 1 4 6 4 1

. . . .

REALNA ŠTEVILA

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^2 + b^2 = \,!$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

POTENCE

$$x^a \cdot x^b = x^{a+b} \;;\; x^a : x^b = x^{a-b} \;;\; (xy)^a = x^a y^a \;;\; \left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} \;;\; (x^a)^b = x^{a \cdot b}$$

$$x^{-a} = \frac{1}{x^a}$$

KORENI

$$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

$$\sqrt[n]{\sqrt[m]{a \cdot b}} = \sqrt[nm]{a \cdot b}$$

$$\sqrt[n]{a} = a^{1/n}$$

$$\boxed{\begin{array}{l} \sqrt[3]{x^3} = x \\ \sqrt{x^2} = |x| \end{array}} \; !$$

③ Poenostavi:

$$\frac{(4x)^{1/2} \cdot x^{2^{2^2}}}{\sqrt[4]{\sqrt{x^9}} \cdot (x^2)^{2^2}} = \frac{\frac{1}{\sqrt{4x}} \cdot x^{16}}{\sqrt[8]{x^9} \cdot x^8} = \frac{2\sqrt{x} \cdot x^{16}}{x^{9/8} \cdot x^8} = \frac{2 \cdot x^{\frac{1}{2}} \cdot x^{16}}{x^{9/8} \cdot x^{64/8}} =$$

$$= \frac{2\, x^{\frac{33}{2}}}{x^{73/8}} = 2x^{\frac{59}{8}}$$

$$\frac{33}{2}\Big/ 4 - \frac{73}{8} = \frac{132}{8} - \frac{73}{8} = \frac{59}{8}$$

$$\boxed{\begin{array}{l} ax^2 + bx + c \\ x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{array}}$$

ENAČBE IN NEENAČBE:

④ Reši kv. enačbe

a) $x^2 + x - 6 = 0$; $(x+3)(x-2) = 0$ $x_1 = -3$; $x_2 = 2$

b) $\frac{1}{2}x^2 - 2x + 3 = 0 / 2$; $x^2 - 4x + 6 = 0$; $D < 0$; ni realnih rešitev

$$x_{1,2} = \frac{4 \pm \sqrt{-8}}{2} = \frac{4 \pm 2\sqrt{-2}}{2} = 2 + i\sqrt{2}$$

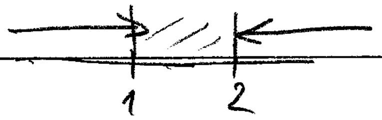
⑤ Poišči množico rešitev naslednjih neenačb:

a) $2x < x+1 < 2x-1$ linearni neenačbi

$2x < x+1$ $\qquad x < 1$

m

$\cap$ $x+1 < 2x-1$ $\qquad 2 < x \Rightarrow x > 2$



PRESEK!

R: $\emptyset$

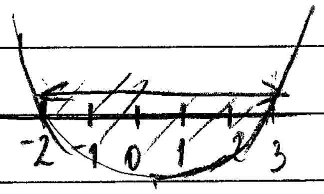
$\boxed{\text{b) } x-5 < 2x-3 \leq x+2 \quad R: (2,5]}$!

c) $x(x-5) < -6$ kvadratna neenačba

$x^2 - 5x + 6 < 0$

$(x-2)(x-3) < 0$ $\qquad x_1 = 2$

$x_2 = 3$



R: $x \in (2,3)$

$\boxed{\text{d) } x^2 \leq 4 \quad R: [-2,2]}$!

⑥ reši enačbe

a) $\sqrt{3x^2 - 7x + 3} = 1 - x \quad /()^2$ ⟸ PREIZKUS

$3x^2 - 7x + 3 = 1 - 2x + x^2$

$2x^2 - 5x + 2 = 0$

$x_{1,2} = \frac{5 \pm \sqrt{25-16}}{4} = \frac{5 \pm 3}{4}$

~~$x_1 = 2$~~

$x_2 = 1/2$ ✓

R: $x = \frac{1}{2}$

b) $\sqrt{2x+1} - \sqrt{x+2} = \sqrt{3x+1} \quad /^2$

$2x+1 - 2\sqrt{2x+1}\sqrt{x+2} + x+2 = 3x+1$

$-2\sqrt{2x+1}\sqrt{x+2} = -2 \quad /(-2)$

$\sqrt{2x+1}\sqrt{x+2} = 1 \quad /^2$

$(2x+1)(x+2) = 1$

$2x^2 + 5x + 1 = 0$

$x_{1,2} = \frac{-5 \pm \sqrt{25-8}}{4} = \frac{-5 \pm \sqrt{17}}{4}$

pogoji za rešitev

$$\Gamma \begin{cases} 2x+1 \geq 0 \rightarrow x \geq -\frac{1}{2} \\ x+2 \geq 0 \rightarrow x \geq -2 \\ 3x+1 \geq 0 \rightarrow x \geq -\frac{1}{3} \end{cases}$$

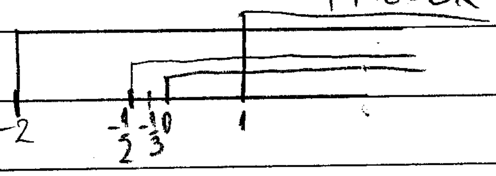
$\sqrt{2x+1} - \sqrt{x+2} \geq 0$

$\sqrt{2x+1} \geq \sqrt{x+2} \quad /^2$

$2x+1 \geq x+2$

$x \geq 1$

PRESEK



R: ni rešitev

⑥ $\sqrt{x+\sqrt{2x-1}}+\sqrt{x-\sqrt{2x-1}}=\sqrt{2}$ R: $[\frac{1}{2},1]$!

⑦ Reši neenačbe

a) $\sqrt{x}\overset{+}{+}\sqrt{x+1}\overset{+}{>}3 \ /^2$ ⇐ PREIZKUS

$x+2\sqrt{x}\sqrt{x+1}+x+1>9$

$2\sqrt{x}\sqrt{x+1}\geq 8-2x$

$\sqrt{x}\sqrt{x+1}>4-x \ /^2$

$x(x+1)\geq 16-8x+x^2$

$\cancel{x^2}+x>16-8x+\cancel{x^2}$

$16>9x$

R: $x\geq\frac{16}{9}$

$x\in(\frac{16}{9},\infty)$

$\boxed{x>\frac{16}{9}}$ kandidati za rešitev

POGOJI

$\begin{cases} x\geq 0 \\ x+1\geq 0 \Rightarrow x\geq -1 \end{cases}$

$x\geq 0$

R: (presek množice kandidatov in pogoja)

b) $\frac{1+x^2}{1-x^2}\leq 1 \ /(1-x^2)^2 \Longrightarrow (1+x^2)(1-x^2)\leq(1-x^2)^2$

$(1+x^2)(1-x^2)-(1-x^2)^2\leq 0$

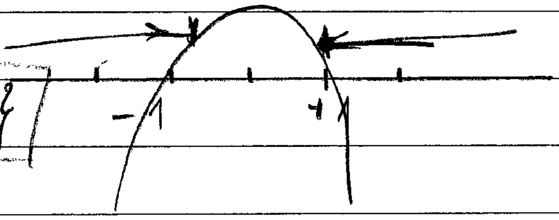
$(1-x^2)((1+x^2)-(1-x^2))\leq 0$

$(1-x^2)(2x^2)\leq 0$

$(1-x^2)\leq 0$ ali $2x^2=0$

$(1-x)(1+x)\leq 0$

$\boxed{x\in(-\infty,-1]\cup[1,\infty)\cup\{0\}}$



pogoji:

$x\neq\pm 1$

R: $x\in(-\infty,-1)\cup(1,\infty)\cup\{0\}$

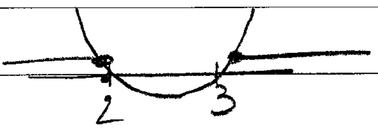
c) $\frac{2x-3}{x-2}\leq 3$ $x\neq 2$ R: $(-\infty,2)\cup[3,\infty)$!

$(2x-3)(x-2)\leq 3(x-2)^2$

$2x^2-4x-3x+6\leq 3x^2-12x+12$

$x^2-5x+6\geq 0$

$(x-2)(x-3)$



$x\in(-\infty,2]\cup[3,\infty)$

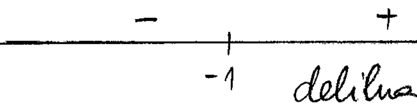
MAT – VAJE

Absolutna vrednost – grafično je razdalja od izhodišča

$|x| = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$

a) $x + |x+1| = 3$ → I. $x+1 \geq 0$; $x \geq -1$

II. $x+1 < 0$; $x < -1$



delilna točka raz

Ralno os na 2 intervala.

I. $x + x + 1 = 3$

$2x = 2$ $\{1\}$

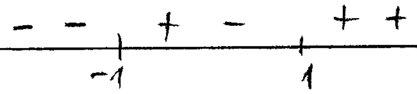
$x = 1$ – resitev ustreza pogoju → $x \geq -1$

II. $x - x - 1 = 3$

$0 \neq 4$ – resitev ne obstaja ! $\{\emptyset\}$

R: $\{1\}$

b) $|x+1| + |x-1| = 2$



I. $x < -1$

$-(x+1) - (x-1) = 2$

$-x - 1 - x + 1 = 2$

$-2x = 2$

$x = -1$ resitev ne ustreza

$\{\emptyset\}$

II. $-1 \leq x < 1$

$x+1 - (x-1) = 2$

$0 = 0$

$[-1, 1)$

III. $x \geq 1$

$x + 1 + x - 1 = 2$

$2x = 2$

$x = 1$ $\{1\}$

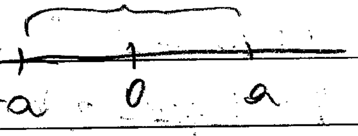
R: $[-1,1) \cup \{1\}$

$[-1,1]$

c) $|x^2 + 3x - 1| < 3$

$-3 < x^2 + 3x - 1 < 3$

$|x| < a$



⇓

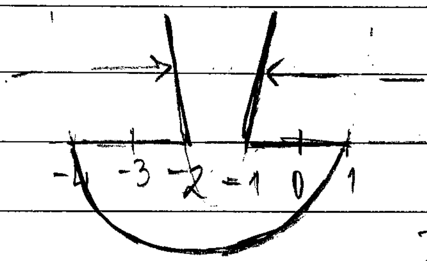
$-a < x < a$

I. $x^2 + 3x - 1 > -3$ II. $x^2 + 3x - 1 < 3$

$x^2 + 3x + 2 > 0$ $x^2 + 3x - 4 < 0$

$(x+1)(x+2) > 0$ $(x+4)(x-1) < 0$

$x_1 = -2$
$x_2 = -1$
$x_3 = -4$
$x_4 = 1$



I. $(-\infty, -2) \cup (-1, \infty)$

II. $(-4, 1)$

R: $(-4, -2) \cup (-1, 1)$

in → ∧ → ∧ →pre

d) $|2|x| - 4| < 2$

- najprej postavimo pogoje za notranjo abs. vrednost.

I. $x \geq 0$

$|2x - 4| < 2$

v primerih z več podprimeri iščemo unijo vrednosti.

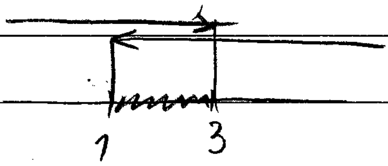
1.) $-2 < 2x - 4 < 2$

$-2 < 2x - 4$ | 2.) $2x - 4 < 2$

$0 < 2x \;/:2$ $\quad 2x < 6$

$0 < x - 1$ $\quad \underline{x < 3}$

$\underline{x > 1}$ $\quad$ R: $(1, 3)$ - ustreza pogoju: $x \geq 0$



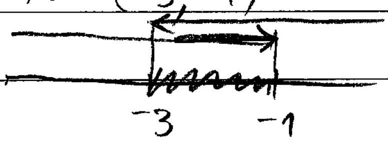
II. $x < 0$

$|-2x - 4| < 2$ $\quad$ 1.) $-2 < -2x - 4$ $\quad$ 2.) $-2x - 4 < 2$

$-2 < -2x - 4 < 2$ $\quad -2x < -2$ $\quad -2x - 6 < 0$

$0 < -x - 1$ $\quad -x - 3 < 0$

R: $(-3, -1)$ $\quad x < -1$ $\quad x > -3$



R: $(-3, -1) \cup (1, 3)$

Matematična indukcija

$1 \cdot 2 + 2 \cdot 3 + \ldots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$

1.) baza indukcije $(n = 1)$ $\quad$ 2.) indukcijski korak $(n-1 \rightarrow n)$

$L: 1 \cdot 2 = 2$ $\quad$ indukcijska predpostavka

$D: \frac{1 \cdot 2 \cdot 3}{3} = 2$ ✓ $\quad (n-1): 1 \cdot 2 + 2 \cdot 3 + \ldots + n(n-1) = \frac{(n-1)\,n \cdot (n+1)}{3}$

$$\underbrace{1\cdot 2 + 2\cdot 3 + \ldots + (n-1)\cdot n}\, + n\cdot(n+1) \overset{\uparrow}{=}$$

ind. predpostavka

$$= \frac{(n-1)n(n+1)}{3} + n(n+1) =$$

$$= \frac{n(n+1)((n-1)+3))}{3} = \frac{n(n+1)(n+2)}{3}$$ ⟸ dokaz indukcijskega koraka

a) $1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

za: $n=1$

$L: 1^2=1$ $\quad D: \frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1$

$n-1 \to n$:

$$1^2+2^2+\ldots+(n-1)^2 = \frac{(n-1)n\cdot(2(n-1)+1)}{6}$$

I.P.

$$1^2+2^2+\ldots+(n-1)^2+n^2 \underset{\overset{\uparrow}{\text{I.P.}}}{=} \frac{(n-1)n(2n-2+1)}{6}+n^2$$

$$= \frac{6n^2+(n-1)n(2n-1)}{6} = \frac{n(6n+(n-1)(2n-1))}{6} =$$

$$= \frac{n(6n+2n^2-n-2n+1)}{6} = \frac{n(6n+2n^2-3n+1)}{6} =$$

$$= \frac{n(2n^2+3n+1)}{6} = \frac{n(n+1)(2n+1)}{6}$$

b) $\frac{9}{(n^3+(n+1)^3+(n+2)^3)}$

I. $n=1$

$1^3+2^3+3^3 = 1+8+27 = 36$

II. $n-1 \to n$

I.P.

$$(n-1)^3+n^3+(n+1)^3 = 9k$$

I.P.

$$n^3+(n+1)^3+(n+2)^3 = n^3+(n+1)^3+n^3+6n^2+12n+8$$

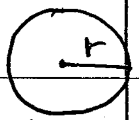
$$= 9k-(n-1)^3+n^3+6n^2+12n+8 = 9k-(n^3-3n^2+3n-1)+(\overset{n^3}{}6n^2+12n+8) =$$

$$= 9k+9n^2+9n+9 = 9\underbrace{(k+n^2+n+1)}_{k'} = 9k'$$

v ravnini

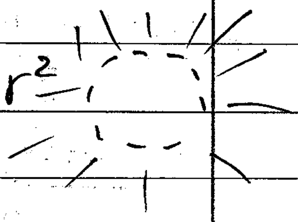
Množice točk v $\mathbb{R}^2$

$x^2+y^2=r^2$



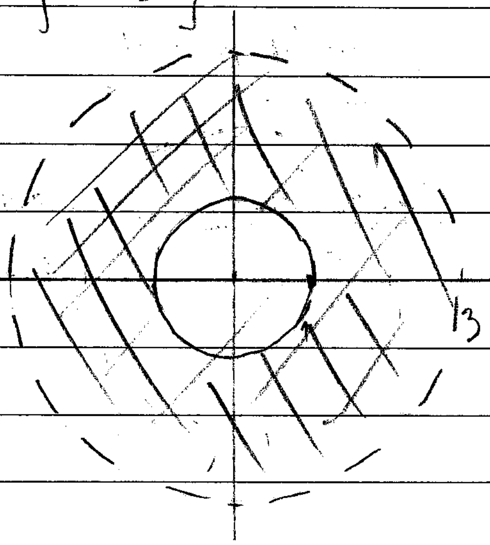
$x^2+y^2<r^2$

$x^2+y^2>r^2$



a) $\{(x,y);\ 1 \leq x^2+y^2 < 9\}$

$1 = x^2+y^3$



b) $\{(x,y);\ y \geq x^2-4,\ (x-1)^2+(y+1)^2 \leq 4\}$

P: $y = x^2-4$

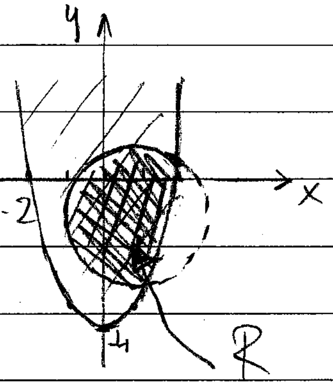
$= (x-2)(x+2)$

$x_1 = 2$

$x_2 = -2$

E: $S(1,-1)$

$r = 2$



$(x-p)^2+(y-q)^2=r^2$

$S(p,q),\ r$

c) $\{(x,y);\ |x|+|y| \leq 1\}$

I. $x+y \leq 1$

$x+y = 1$

$y = -x+1$

II. $-x+y = 1$

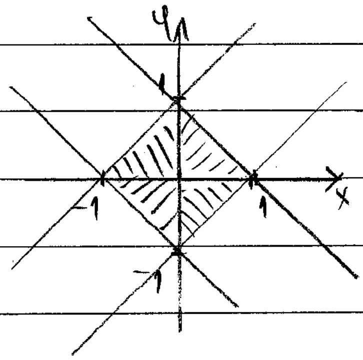
$y = x+1$

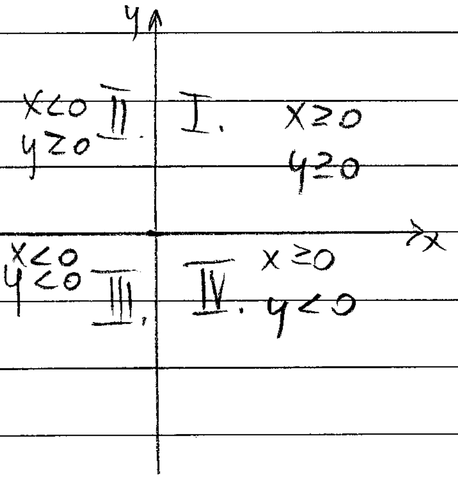
III. $-x-y = 1$

$y = -x-1$

IV. $x-y = 1$

$y = x-1$



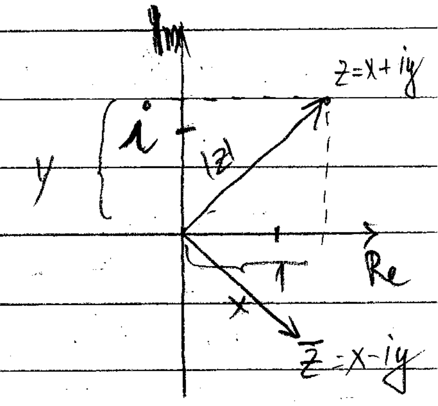


KOMPLEKSNA ŠTEVILA

$\mathbb{C}=\{z=x+iy;\ x,y\in\mathbb{R},\ i^2=-1\}$

$\bar{z}=x-iy$

$|z|=\sqrt{x^2+y^2}=\sqrt{z\cdot\bar{z}}$



$i^1=i$

$i^2=-1 \qquad i^{2010}=i^2=-1$

$i^3=-i \qquad \cancel{i}\ 2010:4=502$

$i^4=1 \qquad\qquad \textcircled{2}$

a) $(1+2i)^2+\frac{25}{3+4i}+\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)-\cancel{2}i^{207}=$

$=1+4i+4i^2+\frac{25(3-4i)}{(3+4i)(3-4i)}+\left(\frac{1}{4}-\frac{i^2\cdot 3}{4}\right)-i^3=$

$=1+4i-4+\frac{((75-100i))}{9+16}+\left(\frac{1}{4}+\frac{3}{4}\right)+i=$

$=-\cancel{3}+\cancel{4i}+\cancel{3}-\cancel{4i}+\underline{\underline{1+i}}$

$\operatorname{Re}w=1$

$\operatorname{Im}w=1$

$\bar{w}=1-i$

$|w|=\sqrt{2}$

b)

$w=\frac{(3+i)(1+i)}{2-i}=$

$=\frac{(3+i)(1+i)(2+i)}{5}=$

$=\frac{(3+\overset{4i}{3i+i}+i^2)(2+i)}{5}=\frac{\cancel{6}+3i+8i+\cancel{4i^2}+\cancel{2i^2}+i^3}{5}=\frac{2i+8i}{5}$

$=\underline{\underline{2i}} \qquad \operatorname{Re}w=0 \qquad \bar{w}=-2i$

$\operatorname{Im}w=2 \qquad |w|=\sqrt{4}=\underline{2}$

c) $\frac{2-3i}{3-i}-\frac{4+i}{3+i}=\frac{(2-3i)(3+i)-(4+i)(3-i)}{(3-i)(3+i)}=$

$=\frac{(9-7i)-(13-i)}{10}=\frac{-4-6i}{10}=-\frac{2}{5}-\frac{3}{5}i$

$\operatorname{Re}w=-\frac{2}{5} \qquad \bar{w}=-\frac{2}{5}+\frac{3}{5}i$

$\operatorname{Im}w=-\frac{3}{5} \qquad |w|=\sqrt{\frac{13}{25}}=\frac{\sqrt{13}}{5}$

d) $w = \frac{z-\bar{z}}{2} = \frac{x+iy - x+iy}{2} = \frac{2iy}{2} = iy$

$\mathrm{Re}\, w = \emptyset$

$\mathrm{Im}\, w = y$

$\bar{w} = -iy$

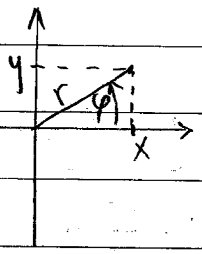
$|w| = \sqrt{0^2+y^2} = |y|$

$z = x+iy$

$$\boxed{\sqrt{x^2} = |x| \; !}$$

POLARNI ZAPIS

$z = x+iy$



$$\boxed{\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ r &= \sqrt{x^2+y^2} \\ \varphi &= \arctan\frac{y}{x} \;\; !!! \end{aligned}}$$

$\Downarrow$

2 možna kota, ker ima $\tan x$ periodo π.

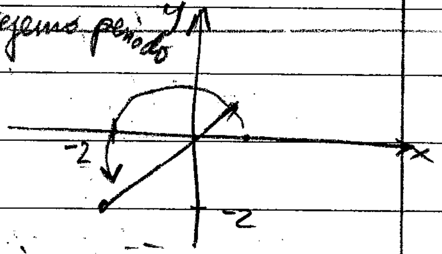
$z = -2-2i$

$x = -2$

$y = -2$

$\varphi = \arctan 1 = \frac{\pi}{4} + \pi$ } prištejemo periodo

$r = \sqrt{8} = 2\sqrt{2}$ $\qquad = \frac{5\pi}{4}$



φ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\varphi$	$0 \; \frac{\sqrt{0}}{2}$	$1/2$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$1 = \frac{\sqrt{4}}{2}$
$\cos\varphi$	1	$\sqrt{3}/2$	$\frac{\sqrt{2}}{2}$	$1/2$	0
$\tan\varphi$	$\emptyset$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞

→ koreni naraščajo

→ koreni padajo

$z = \sqrt{8}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

MAT - VAJE

DN: $\left|\frac{x}{x+4}\right| < 1$ R: $(-2,\infty)$

c) $|2|x|-4| < 2$

$-2x-4<0$

$x\in(-2,-1)$

$x \geq 0$

$x<0$

$|2x-4|<2$

$|-2x-4|<2$

$-2x-4 \geq 0$

$2x-4 \geq 0$

$2x-4<0$

$-2x-4<2$

$2x-4<2$

$-2x+4<2$

$x>-3$

$x<0$

$x\leq -2$

$x\in[-3,2]$

$x<3$

$x\geq 0$

$x \geq 2$

PRESEK $x\in[2,3)$

$-2x<-2$ $/(-2)$

~~-2x-~~

$x>1$

$x\geq 0$

$x<2$

PRESEK $x\in(1,2)$

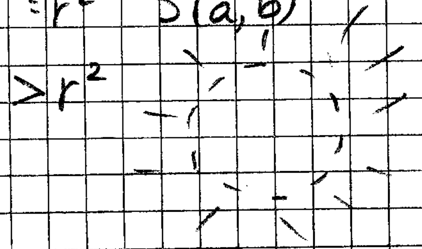
R: $(1,3)\cup(-3,-1)$

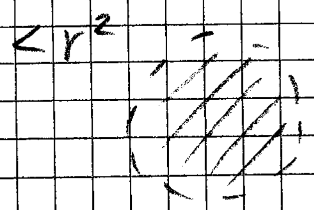
MNOŽICE V RAVNINI

krivulje 2. reda:

- KROŽNICA: $x^2+y^2=r^2$ $S(0,0)$, r = polmer

$(x-a)^2+(y-b)^2=r^2$ $S(a,b)$

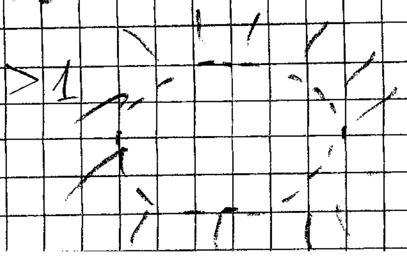


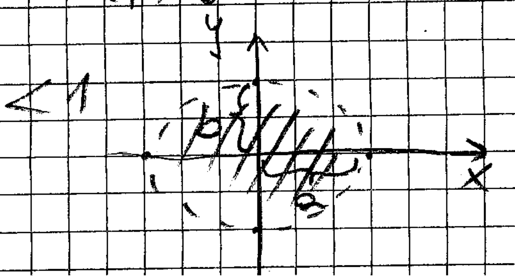


- ELIPSA:

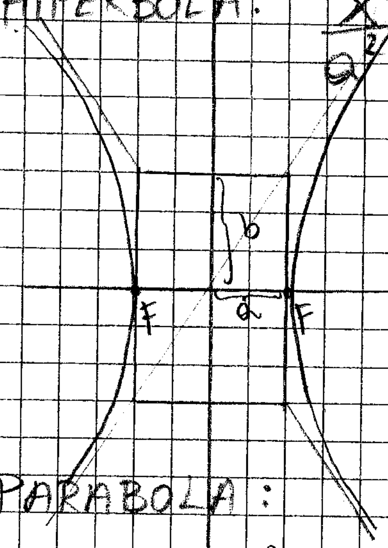
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a,b => polosi $S(0,0)$

$\frac{(x-p)^2}{a^2}+\frac{(y-q)^2}{b^2}$ $S(p,q)$

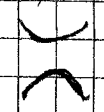




HIPERBOLA: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; a, b ⇒ polosi $S(0,0)$

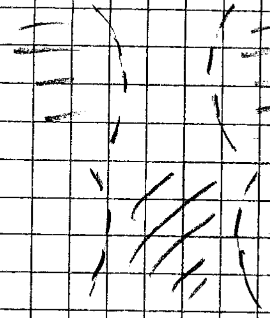


$=-1$



>1

<1

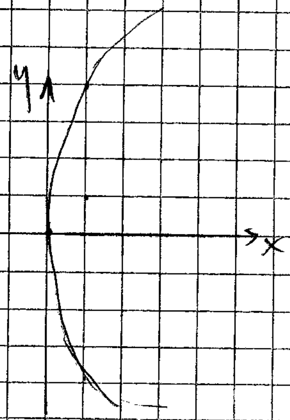


PARABOLA:

$y=ax^2+bx+c$

$y^2=px$

$p<0$ $p>0$

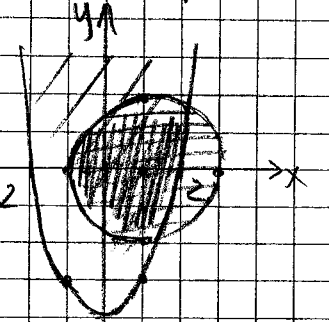


1. Skiciraj podmnožice realne ravnine

a) $\{(x,y);\ y\geq x^2-4,\ (x-1)^2+(y+1)^2\leq 4\}$

$(x-2)(x+2)$
$T(0,-4)$

$S(1,-1)$
$r=2$



DN
$\{(x,y);\ y^2\leq 2x+1\}$

I. $x+y\geq 0$
$x+y\leq 2$
$y\leq -x+2$
$y\geq -x$

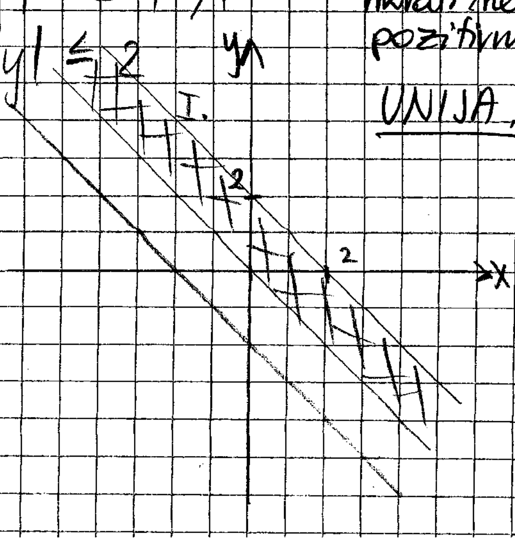
II. $x+y<0$
$-(x+y)\leq 2$
$-x-y\leq 2$
$y\geq -2-x$
$y<0$

c) $\{(x,y);\ x^2+2xy+y^2\leq 4\}$

$x^2+2xy+y^2\leq 4$

$(x+y)^2\leq 4\ /\sqrt{\ }$

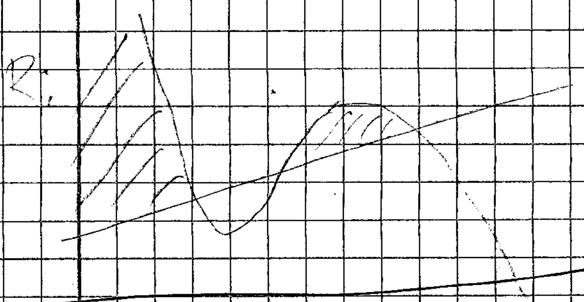
$|x+y|\leq 2$



pri abs. vrednosti ni preseka, ker ~~se~~ funkcije hkrati ne more biti pozitivna in negativna.

<u>UNIJA!</u>

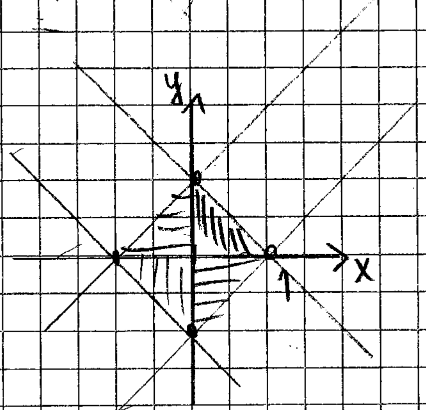
d) $\{(x,y);\ 2+x-2x^3-x^3 \geq y \wedge y \geq \frac{x}{2}+1\}$ DN



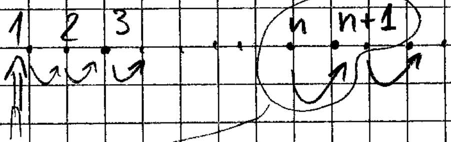
e) $\{x,y;\ |x|+|y| \leq 1\}$

$|x|+|y| \leq 1$

I.	II. 2kV	III. 4.kV	IV 3kV
$x \geq 0$ $y \geq 0$ 1.kv.	$x<0$ $y \geq 0$	$x \geq 0$ $y<0$	$x<0$ $y<0$
$x+y \leq 1$	$-x+y \leq 1$	$y \geq x-1$ nad	$-x-y \leq 1$
$y \leq 1-x$ pod	$y \leq x+1$ pod		$y \geq x-1$



MATEMATIČNA INDUKCIJA:



Velja samo za naravna števila!

BAZA INDUKCIJE:

Dokažemo, da trditev velja za neko začetno naravno število npr.: 1.

INDUKCIJSKI KORAK → ind. predpostavka

Predpostavimo, da trditev velja za n in dokažemo, da tedaj velja tudi za naslednika → $n+1$.

Z indukcijo dokaži trditve:

$1+2+2^2+\ldots+2^{n-1}=2^n-1$

BAZA: $n=1$

$L=1$
$D=1$

IND. KORAK

$n \to n+1$

IND. PREDPOSTAVKA

$1+2+2^2+\ldots+2^{n-1}=2^n-1$

$1+2+2^2+\ldots+2^{(n+1)-1}=2^{(n+1)}-1$

$1+2+2^2+\ldots+2^n=2^{n+1}-1$

$\underbrace{1+2+2^2+\ldots+2^{n-1}}+2^n=2^{n+1}-1$

$2^n-1+2^n=2\cdot 2^n-1=$

$=2^{n+1}-1$

DN:

$1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$ | DN

c) $9 \,\|\, \left(n^3+(n+1)^3+(n+2)^3\right)$

(deli)

$n^3+(n+1)^3+(n+2)^3=9\cdot k \;, k\in\mathbb{N}$

BAZA

$n=1$

$1+8+27=9k$

$36=9k$ ✓

$k=4$

IND. KORAK

$\boxed{n\to n+1}$ IND. PREDPOSTAVKA: $n^3+(n+1)^3+(n+2)^3=9k,\; k\in\mathbb{N}$

ali velja $(n+1)^3+(n+2)^3+(n+3)^3=9\cdot l,\; l\in\mathbb{N}$

$\boxed{(n+1)^3+(n+2)^3+n^3+9n^2+27n+27}$

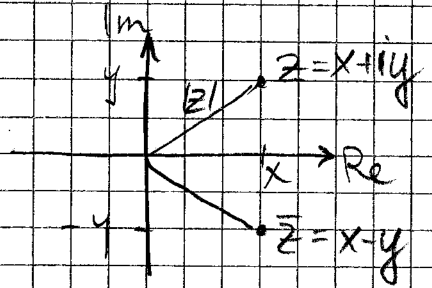
$9k+9n^2+27n+27=9(\underbrace{k+n^3+3n+3}_{l})$

KOMPLEKSNA ŠTEVILA

$z=x+iy \qquad i=\sqrt{-1}\;;\; i^2=-1$

$x,y\in\mathbb{R}$

$x\in Re(z)$ realna
$y\in Im(z)$ imaginarna } komponenta



$|z|=\sqrt{x^2+y^2}=\sqrt{z\,|z|}$

Poenostavi:

1. $i^{2010}=i^2=-1 \qquad 2010:4=502+$ ost. 2
10

2. $\left(\frac{1-i}{3-2i}\right)^2=\frac{1-2i-1}{9-12i-4}=\frac{-2i}{(5-12i)}\frac{(5+12i)}{(5+12i)}=\frac{-10i+24}{25+144}=\frac{-10i+24}{169}$

$Re(z)\,\frac{+24}{169} \qquad Im(z)=\frac{-10}{169} \qquad |z|=\sqrt{\left(\frac{24}{169}\right)^2+\left(-\frac{10}{169}\right)^2}=\frac{2}{13}$

3. $\frac{2-3i}{3-i}-\frac{4+i}{3+i}=\frac{(2-3i)(3+i)-(4+i)(3-i)}{(3-i)(3+i)}=\frac{6+2i-9i+3-(12-4i+3i+1)}{10}=$

$=\frac{9-7i-13+i}{10}=\frac{-4+6i}{10}=-\frac{4}{10}+\frac{6i}{10} \qquad |z|=\sqrt{\frac{4}{25}+\frac{9}{25}}=\frac{\sqrt{13}}{5}$

DN

$\frac{(3+i)(1+i)}{2-i} \qquad R: 2i$

$\frac{z-\bar{z}}{2}=\frac{(x+iy)-(x-iy)}{2}=\frac{i2y}{2}=iy \qquad Re(z)=0$

$z=x+iy \qquad |z|=\sqrt{0+y^2}=|y| \qquad Im=y$

MATEMATIČNA INDUKCIJA

$$1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

$L=1^2=1$ $\quad$ $L=D$

$$D=\frac{1\cdot 2\cdot 3}{6}=1$$

predpostavimo, da velja za n

I.P. $1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$

dokazujem, da velja za $n+1$

$$1^2+2^2+\ldots+n^2+(n+1)^2=\frac{(n+1)(n+2)(2n+3)}{6}$$

$$L:\ \underbrace{1^2+2^2+\ldots+n^2}+(n+1)^2 \overset{\uparrow}{\underset{I.P.}{=}} \quad -\|-$$

$$\frac{n(n+1)(2n+1)}{6}+(n+1)^2=\frac{(n+1)(n+2)(2n+3)}{6}$$

$$\frac{n(n+1)(2n+1)}{6}+\frac{6(n+1)^2}{6}=\frac{(n+1)(n+2)(2n+3)}{6}$$

$$\frac{(n+1)\big(n(2n+1)+6(n+1)\big)}{6}=\frac{(n+1)(2n^2+7n+6)}{6}=\frac{(n+1)(n+2)(2n+3)}{6}$$

I. $\sqrt{x+\sqrt{2x-1}}+\sqrt{x-\sqrt{2x-1}}=\sqrt{2}\ \Big/^2$

$$x+\cancel{\sqrt{2x-1}}+2\sqrt{(x+\sqrt{2x-1})(x-\sqrt{2x-1})}+x-\cancel{\sqrt{2x-1}}=2$$

$$2x+2\sqrt{(x+\sqrt{2x-1})(x-\sqrt{2x-1})}=2$$

$$2x+2\sqrt{x^2-(2x-1)}=2$$

$$2\sqrt{x^2-(2x-1)}=2-2x\ \Big/:2$$

$$\sqrt{x^2-2x+1}=1-x\ \Big/^2$$

$$x^2-2x+1=1-2x+x^2$$

$$0=0$$

$\sqrt{x^2}=|x|$

II. $\sqrt{(x-1)^2}=1-x$

$|x-1|=1-x$

~~4~~ 1. $x-1\geq 0$

$x\geq 1$

$x-1=1-x$

$2x=2$

$x=1$

2. $x-1<1$

$-x+1=1-x$

$0=0$

$(-\infty,1)$

III. $2x-1\geq 0$

$2x\geq 1$

$x\geq\frac{1}{2}$ $\quad$ $\left[\frac{1}{2},1\right]$

$\{x;\ \sqrt{\sqrt{8-2x}-x} \geq 1\}$

$\sqrt{\sqrt{8-2x}-x} \geq 1 \ /^2$

$\sqrt{8-2x}-x \geq 1 \ /^2$

$8-2x \geq x^2+2x+1$

$-x^2-4x+7 \geq 0 \ /(-1)$

$x^2+4x-7 \leq 0$

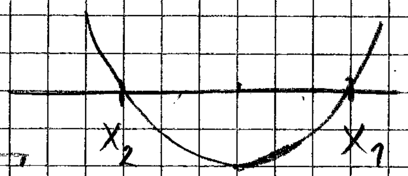
I. $D = 16+28 = 44$

$x_{1,2} = \frac{-4 \pm \sqrt{44}}{2} = \frac{-4 \pm 2\sqrt{11}}{2}$

$x_1 = -2 + \sqrt{11}$

$x_2 = -2 - \sqrt{11}$

$x \in [-2-\sqrt{11}, 2+\sqrt{11}]$



II. $8-2x \geq 0$

$8 \geq 2x$

$x \leq 4$

III. $\sqrt{8-2x}-x \geq 0$

$\sqrt{8-2x} \geq x \ /^2$

$8-2x \geq x^2$

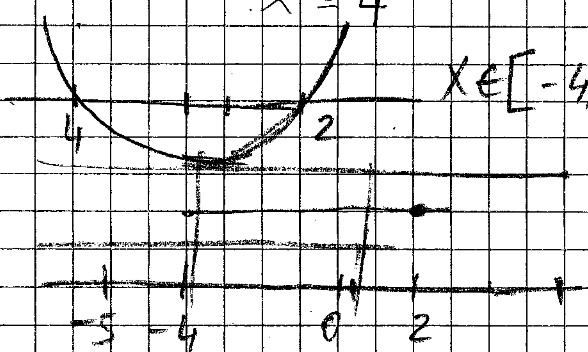
$x^2+2x-8 \leq 0$

$(x+4)(x-2) \leq 0$

$x = -4$

$x = 2$

$x \in [-4, 2]$



$x \in [-4, -2+\sqrt{11}]$

$|2-\sqrt{x}| < 2$

I. $2-\sqrt{x} \geq 0$

$2-\sqrt{x} < 2$

$-\sqrt{x} < 0$

$\sqrt{x} > 0 \ /^2$

$x > 0$

$-\sqrt{x} \geq -2$

$\sqrt{x} \leq 2 \ /^2$

$x \leq 4$

$R: (0, 4]$

II. $2-\sqrt{x} < 0$

$\sqrt{x} > 2 \ /^2$

$x > 4$

$-2+\sqrt{x} < 2$

$\sqrt{x} < 4 \ /^2$

$x < 16$

$R: (4, 16)$

$R: (0,4] \cup (4,16)$

$R: (0,16)$

$\{(x,y);\ -2 \leq x \leq 0,\ |y| \geq |x|\}$

I. $x \geq 0,\ y \geq 0$

$|y| \geq |x|$

$y \geq x$

II. $x < 0,\ y \geq 0$

$y \geq (-x)$

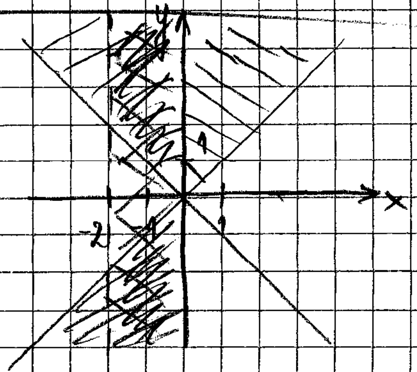
III. $x < 0,\ y < 0$

$-y \geq -x$

$y \leq x$

IV. $x \geq 0,\ y < 0$

$y \leq -x$



$\{(x,y);(y-1)(x^2+y^2-1)=0,\ x^3+xy^2-5x=0,\ y-|x|+1=0\}$

I. $(y-1)(x^2+y^2-1)=0$

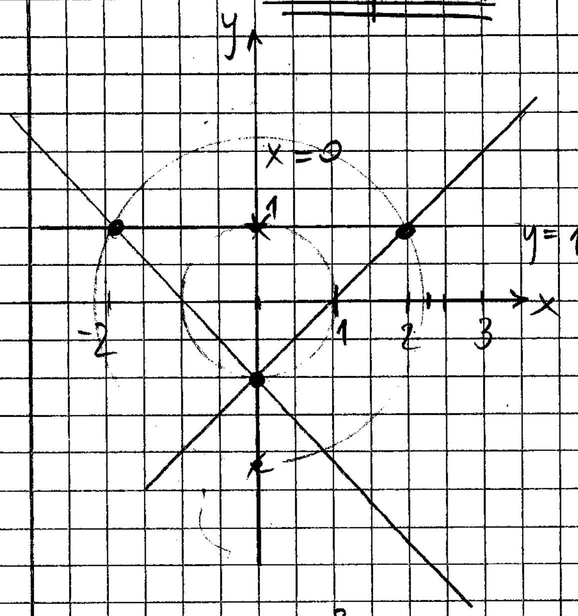
$y=1$

$\underline{\underline{x^2+y^2=1}}$

II. $x^3+xy^2-5x=0$

$x(x^2+y^2-5)=0$

$x=0$ $\quad \underbrace{x^2+y^2=5}$



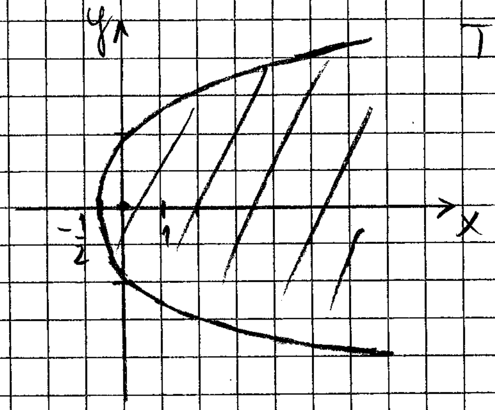
III. $y-|x|+1=0$

$y=|x|-1$

$\{(0,-1),(-2,1),(2,1)\}$

$\{(x,y);\ y^2\leq 2x+1\}$

$y^2\leq 2x+1$



$T(0,0)$

$0\leq 1$ ✓

$y=\sqrt{2x+1}$

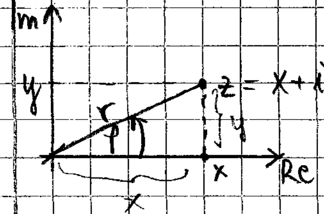
$x^2=2y+1$

$y=\frac{1}{2}x^2-\frac{1}{2}$

MAT. VAJE

1. KOLOKVIJ: 26. november, 18h

polarni zapis kompleksnega števila



φ – $\overset{\text{kot}}{\text{vedno}}$ merimo levo od realne osi.

x, y : kartezični koordinati

r, φ : polarni koordinati

$$\left.\begin{array}{l}\sin\varphi = \frac{y}{r} \Rightarrow \boxed{y = r\sin\varphi}\\ \cos\varphi = \frac{x}{r} \Rightarrow \boxed{x = r\cos\varphi}\end{array}\right\} \quad r,\varphi \to x,y \qquad tg = \frac{y}{x}$$

$$\left.\begin{array}{l}\boxed{r = \sqrt{x^2+y^2}}\\ \boxed{\varphi = \arctan\frac{y}{x}}\end{array}\right\} \quad x,y \to r,\varphi$$

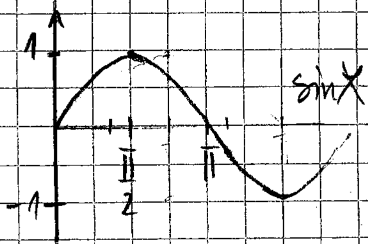
$z = x + iy$

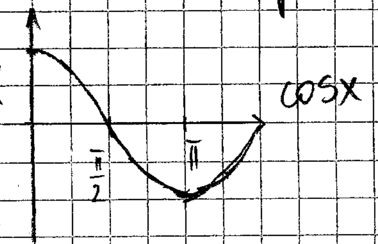
$\boxed{z = r\cos\varphi + ir\sin\varphi = r(\cos\varphi + i\sin\varphi)}$ ⇒ polarni zapis kompleksnega števila

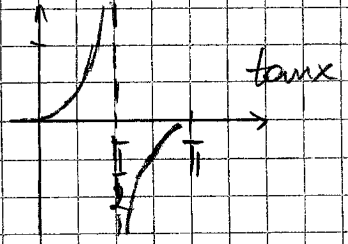
$e^{i\varphi} = \cos\varphi + i\sin\varphi$ ⟹ Eulerjeva formula

$\boxed{z = r(\cos\varphi + i\sin\varphi) = r\cdot e^{i\varphi}}$ ⇒ pol. zapis. kompl. št.

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin\varphi$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos\varphi$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	
$\tan\varphi$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$\pm\infty$ ↑ pol				

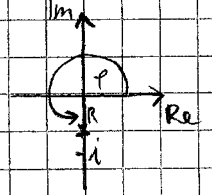






1. zapiši v polarni obliki

a) $z = \frac{1-i}{1+i}\cdot\frac{1-i}{1-i} = \frac{1-2i+i^2}{1+1} = \frac{-2i}{2} = -i$

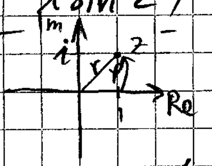


$r = 1$

$\varphi = \frac{3\pi}{2}$

$z = -i = r(\cos\varphi + i\sin\varphi) = 1\cdot\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

b) $z = 1+i$



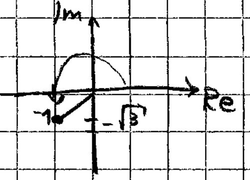
$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\varphi = \arctan\frac{1}{1} = \arctan 1$; $\varphi = \frac{\pi}{4}$; $z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

c) $z = \cancel{-1 i\sqrt{3}} = -1 - i\sqrt{3}$ $\quad x = -1$, $y = -\sqrt{3}$ $\quad z = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$r = \sqrt{1+3} = \sqrt{4} = 2$

$\varphi = \arctan\frac{-\sqrt{3}}{-1} = \arctan\sqrt{3} \Rightarrow \varphi = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$



$\boxed{\underline{\overline{III}.\text{ in }\overline{IV}.\text{ kv., ker ima tangens periodo }k\pi!}}$

d) $\boxed{\begin{array}{l} z = -2-2i \\ R: z = 2\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) \end{array}}$ DN!

2.) Poenostavi (potenciraj / koreni) kompl. št.

a) $\left(-\sqrt{3}+3i\right)^7$ $\Rightarrow$ pretvorimo v polarno obliko in uporabimo DeMoivrovo formule:

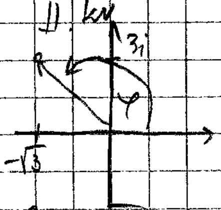
$$\left(r(\cos\varphi + i\sin\varphi)\right)^n = r^n(\cos n\varphi + i\sin n\varphi) \qquad n \in \mathbb{N}$$

$$\sqrt[n]{r(\cos\varphi + i\sin\varphi)} = \sqrt[n]{r}\left(\cos\left(\frac{\varphi+2k\pi}{n}\right) + i\sin\left(\frac{\varphi+2k\pi}{n}\right)\right) \qquad k \in \mathbb{Z}\ (0, 1, \dots, n-1$$

$z = \left(-\sqrt{3}+3i\right)^7$

$r = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$

$\varphi = \arctan -\frac{3}{\sqrt{3}} = -\frac{3\sqrt{3}}{3} = \arctan -\sqrt{3} = \varphi = \frac{2\pi}{3}$



$z = 2\sqrt{3}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$\left(2\sqrt{3}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right)^7 = \left(2\sqrt{3}\right)^7\left(\cos\frac{14\pi}{3} + i\sin\frac{14\pi}{3}\right)$

$2^7 3^3\sqrt{3}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$2^7 3^3\sqrt{3}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{\cancel{2}} \cdot 2^{\cancel{7}\,6} \cdot 3^3\sqrt{3}\left(-1+i\sqrt{3}\right)$

$\frac{14\pi}{3} - \frac{4\pi}{3} \cdot 3 = \frac{2\pi}{3}$ zaradi periodičnosti

$2^6 3^3\sqrt{3}\left(-1+i\sqrt{3}\right)$

$= \boxed{-2^6 3^3\sqrt{3} + i\,2^6 3^4}$

b) $\left(\frac{1}{2} - \frac{i}{2}\right)^{18} =$ $\boxed{IV.\text{ kv}}$

$z = \frac{1}{2} - \frac{i}{2}$

$r = \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{\sqrt{2}}{2}$

$\varphi = \arctan\frac{-\frac{1}{2}}{\frac{1}{2}} = \arctan(-1) = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$

$\frac{63}{2}\pi - \frac{60}{2} = \frac{3\pi}{2}$ $\qquad \underline{\underline{k \cdot 2\pi}}$

$z = \frac{\sqrt{2}}{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

$\left(\frac{\sqrt{2}}{2}\right)^{18}\left(\cos\frac{126\pi}{4} + i\sin\frac{126\pi}{4}\right) = \frac{2^9}{2^{18}}\left(\cos\frac{63}{2}\pi + i\sin\frac{63}{2}\pi\right) = 2^{-9}(0-i) = -i2^{-9} = \underline{\underline{-\frac{i}{512}}}$

c) $\sqrt[4]{1+i}$ $\qquad z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z = 1+i$

$r = \sqrt{2}$

$\varphi = \frac{\pi}{4}$

$$\sqrt[4]{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)} = \sqrt[8]{2}\cdot\left(\cos\frac{\frac{\pi}{4}+2k\pi}{4} + i\sin\frac{\frac{\pi}{4}+2k\pi}{4}\right)$$

$k = 0, 1, 2, 3$

$k = 0 \quad \sqrt[8]{2}\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right)$ $\quad + \frac{8\pi}{16}$

$k = 1 \quad \sqrt[8]{2}\left(\cos\frac{\frac{\pi}{4}+2\pi}{4} + i\sin\frac{\frac{\pi}{4}+2\pi}{4}\right) = \sqrt[8]{2}\left(\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right)$ $\quad + \frac{8\pi}{16}$

$k = 2 \quad \sqrt[8]{2}\left(\cos\frac{\frac{\pi}{4}+4\pi}{4} + i\sin\frac{\frac{\pi}{4}+4\pi}{4}\right) = \sqrt[8]{2}\left(\cos\frac{17\pi}{16} + i\sin\frac{17\pi}{16}\right)$ $\quad + \frac{8\pi}{16}$

$k = 3 \quad \sqrt[8]{2}\left(\cos\frac{\frac{\pi}{4}+6\pi}{4} + i\sin\frac{\frac{\pi}{4}+6\pi}{4}\right) = \sqrt[8]{2}\left(\cos\frac{25\pi}{16} + i\sin\frac{25\pi}{16}\right)$

③ Resi ciklometricne enacbe

a) $z\cdot\bar{z} = -1$ → za velike potence in korene

$z = x+iy = r(\cos\varphi + i\sin\varphi)$

$(x+iy)(x-iy) = -1$

$x^2+y^2 = -1 \qquad \boxed{x,y\in\mathbb{R}} \quad R: \emptyset$

ni realnih resitev!

b) $\boxed{\begin{array}{l}|z| + z = 2+i \\ z = \frac{3}{4}+i\end{array}}$ DN!

c) $2z^2 - 3\bar{z}^2 = 10i \quad ; \quad z = x+iy$

$2(x+iy)^2 - 3(x-iy)^2 = 10i$

$2(x^2+2ixy-y^2) - 3(x^2-2ixy-y^2) = 10i$

$2x^2+4ixy-2y^2-3x^2+6ixy+3y^2 = 10i$

$-x^2+10ixy+y^2 = 10i+0$

$\mathrm{Re}: -x^2+y^2 = 0 \;/(-1)$

$\mathrm{Im}: \cancel{10}xy = \cancel{10}$

$xy = 1$

$x = \frac{1}{y}$

$y^2 - \left(\frac{1}{y}\right)^2 = 0$

$\frac{y^4-1}{y^2} = 0$

$y^4-1 = 0$

$y^4 = 1$

$(y^2-1)(y^2+1) = 0$

$(y-1)(y+1)(y^2+1) = 0$

$\boxed{\begin{array}{l}y_1 = 1 \\ y_2 = -1\end{array}}$

$x_1 = \frac{1}{y_1} = \frac{1}{1} = \underline{\underline{1}}$

$x_2 = \frac{1}{y_2} = -\frac{1}{1} = \underline{\underline{-1}}$

$\boxed{\begin{array}{l}z_1 = 1+i \\ z_2 = -1-i\end{array}}$

④ Reši enačbe z uporabo polarnega zapisa

a) $z^3 = 1 \,/\sqrt[3]{\ }$ $\quad z_1 = 1$

n rešitev!

$\underline{n = 3}$

$z = \sqrt[3]{1 \cdot (\cos 0 + i \sin 0)}$

$z = r(\cos\varphi + i\sin\varphi)$ $\quad r^3(\cos 3\varphi + i \sin 3\varphi) = \underline{1}(\cos 0 + i \sin 0)$

$r_1 = r_2$

$\varphi_1 = \varphi_2 + k \cdot 2\pi$

$r:\ r^3 = 1 \Rightarrow r = 1$

$\varphi:\ 3\varphi = 0 + 2k\pi \Rightarrow \varphi = \frac{2k\pi}{3};\ k = 0,1,2$

$z_1 = 1 \cdot (\cos 0 + i \sin 0) = \underline{1}$

$z_2 = 1 \cdot \left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}\right) = 1\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = \underline{\underline{-\frac{1}{2} + \frac{i\sqrt{3}}{2}}}$

$z_2 = 1\left(\cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}\right) = 1\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) = \underline{\underline{-\frac{1}{2} - \frac{i\sqrt{3}}{2}}}$

b) $\boxed{\begin{array}{l} z^3 = i \\ z_1 = \frac{\sqrt{3}}{2} + \frac{1}{2}i \\ z_2 = \frac{\sqrt{3}}{2} - \frac{1}{2}i \\ z_3 = -i \end{array}}$ DN!

c) $z^4 = -8 + 8i\sqrt{3}$

$x = -8$

$y = 8\sqrt{3}$

$r = \sqrt{8^2 + (8\sqrt{3})^2} = \sqrt{8^2(1+3)} = 8 \cdot 2 = 16$

$\varphi = \arctan\left(-\frac{8\sqrt{3}}{8}\right) = \arctan(-\sqrt{3}) \Rightarrow \varphi = \frac{2\pi}{3}\ \ \pi \cdot k$

$16\left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}\right) = r^4(\cos(4\varphi) + i \sin(4\varphi))$

$z = r(\cos\varphi + i\sin\varphi)$ $\quad r_1 = r_2$

$\varphi_1 = \varphi_2 + 2k\pi$

$16 = r^4 \Rightarrow r = 2$

$\frac{2\pi}{3} = 4\varphi + 2k\pi \Rightarrow \varphi = \frac{2\pi}{12} - \frac{k\pi}{2}$ $\quad \underline{k = 0,1,2,3}$

$k_0 = 0$ $\quad \varphi = \frac{2\pi}{12} = \frac{\pi}{6}$; $\quad k_1 = 1\ \ \varphi = \frac{2\pi}{12} - \frac{\pi}{2} = \frac{2\pi - 6\pi}{12} = -\frac{4\pi}{12} = -\frac{\pi}{3}$

$k_2 = \varphi = \frac{2\pi}{12} - \pi = -\frac{5\pi}{6}$; $\quad k_3 = 3 \quad \varphi = -\frac{9\pi}{6} = -\frac{3\pi}{2}$

d) $2z^3 = (-1 - i\sqrt{3})^6$ → v polarne, nato potenciramo

$z = 2\sqrt[3]{4}\left(\cos\frac{2k\pi}{6} + i \sin\frac{2k\pi}{6}\right),\ k = 0,1,2$

DN!

⑤ Reši sisteme enačb

a) $|z - 2| = 3$

$|z + 1| = 3$

$z = x + iy$

$|x + iy - 2| = 3$

$|x + iy + 1| = 3$

$|x + iy - 2| = |x + iy + 1| = 3$

$\sqrt{(x-2)^2 + y^2} = \sqrt{(x+1)^2 + y^2} = 3 \,/^2$

$(x-2)^2 + \cancel{y^2} = (x+1)^2 + \cancel{y^2} = 9$

$((x-2) - (x+1))((x-2) + (x+1)) = 0$

$-3(2x-1) = 0$

$x = \frac{1}{2}$

$9 = \left(\frac{3}{2}\right)^2 + y^2 \Rightarrow y^2 = 9 - \frac{9}{4}$

$y^2 = \frac{27}{4}$

$\underline{\underline{y = \pm\frac{3\sqrt{3}}{2}}}$

b) $z_1 \cdot \overline{z_2} = \sqrt{2}$

$\frac{z_1}{z_2} = i\sqrt{2}$

R: $z_2 = \frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}$ DN!

$z_1 = 1 + i$

$z_2 = -1 - i$

c) $(2+i)z_1 + (2-i)z_2 = 6 \quad / (3+2i)$

$(3+2i)z_1 + (3-2i)z_2 = 8 \quad / (2+i)$

$\cancel{(2+i)(3+2i)z_1} + (2-i)(3+2i)z_2 = 6(3+2i)$

$\cancel{(2+i)(3+2i)z_1} + (2+i)(3-2i)z_2 = 8(2+i)$) −

$(2-i)(3+2i)z_2 - (2+i)(3-2i)z_2 = 6(3+2i) - 8(2+i)$

$z_2((2-1)(3+2i) - (2+i)(3-2i)) = 18 + 12i - 16 - 8i$

$z_2((\cancel{6}+i+\cancel{2}) - (\cancel{6}-i+\cancel{2})) = 2+4i$

$z_2 = \frac{2+4i}{2i} = \frac{\cancel{2}(1+2i)}{\cancel{2}i} = \frac{(1+2i)}{i} / \frac{(-i)}{(-i)} = \boxed{\boxed{2-i} = z_2}$

$z_1 = \frac{6-(2-i)z_2}{2+i} = \frac{6-(2-i)(2-i)}{(2+i)}$

$z_1 = \frac{6-(4-4i-1)}{2+i} = \frac{3+4i}{2+i} / \frac{2-i}{2-i}$

$\boxed{z_1} = \frac{10+5i}{5} = \boxed{2+i}$

6. Nariši ~~p~~množice kompleksne ravnine:

a) $\mathrm{Re}(z) + \mathrm{Im}(z^2) = 2$

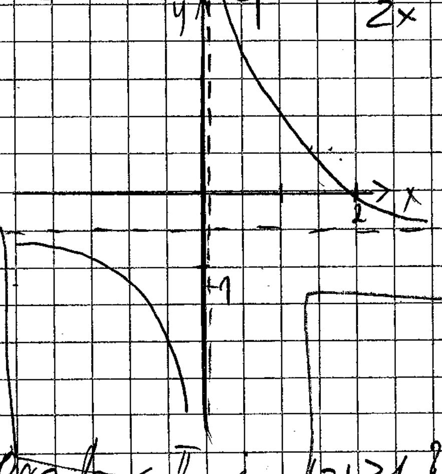
$z = x + iy \Rightarrow z^2 = x^2 + 2ixy - y^2$

$x + 2xy = 2 \Rightarrow y = \frac{2-x}{2x}$

n: $2 - x = 0 \Rightarrow x = 2$

p: $2x = 0 \Rightarrow x = 0$

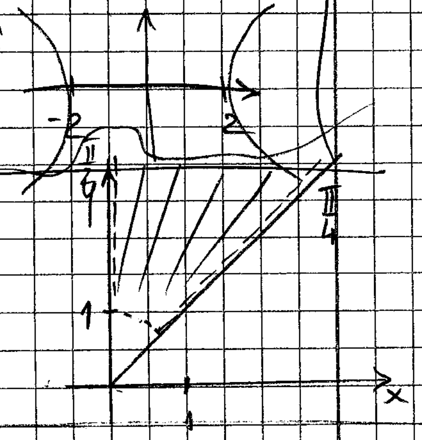
a: $y = -\frac{1}{2}$



DN!

$\mathrm{Re}(z^2) = 4$

R:



c) $\{ z \in \mathbb{C}; \frac{\pi}{4} < \underbrace{\arg z}_{\varphi} < \frac{\pi}{2} \text{ in } |z| > 1 \}$

d) $z \cdot \overline{z} + (1-i)z + (1+i)\overline{z} = 4$

$z = x + iy$

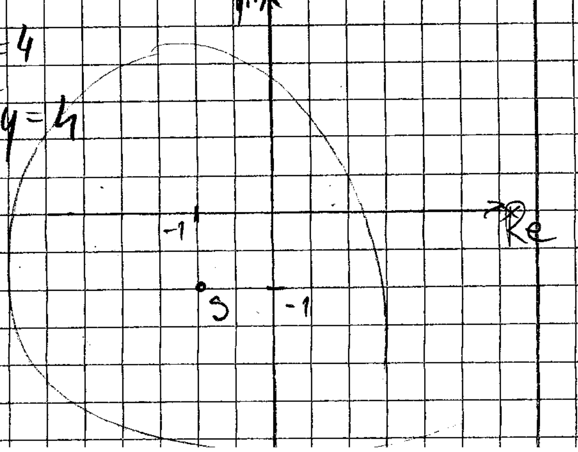
$x^2 + y^2 + (1-i)(x+iy) + (1+i)(x-iy) = 4$

$x^2 + y^2 + \cancel{x+iy} - \cancel{ix}+y + x \cancel{-iy} + \cancel{ix} + y = 4$

$x^2 + y^2 + 2x + 2y = 4$

$(x+1)^2 - 1 + (y+1)^2 - 1 = 4$

$(x+1)^2 + (y^2-1)^2 = 6$



MAT - VAJE

$(-\sqrt{3}+3i)^7$

DeMoivrova formula:

$$\left.\begin{aligned}(r(\cos\varphi+\sin\varphi))^n &= r^n(\cos(n\varphi)+i\sin(n\varphi))\\ \sqrt[n]{r(\cos\varphi+i\sin\varphi)} &= \sqrt[n]{r}\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right)\end{aligned}\right\} k\in\mathbb{Z}$$

$$-\sqrt{3}+3i = r(\cos\varphi+i\sin\varphi)$$

$$(2\sqrt{3})\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$

$$r=\sqrt{3+9}=2\sqrt{3}$$

$$\varphi=\arctan\frac{+3}{-\sqrt{3}}=\operatorname{atan}(-\sqrt{3})=\frac{2\pi}{3}$$

$$(-\sqrt{3}+3i)^7=(2\sqrt{3})^7\left(\cos\left(7\cdot\frac{2\pi}{3}\right)+\left(i\sin 7\cdot\frac{2\pi}{3}\right)\right)=(128\cdot 27\sqrt{3})\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$

$$(3456\cdot\sqrt{3})\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$\left(\frac{1}{2}-i\frac{1}{2}\right)^{18}$$

$$\left(\frac{1}{2}-i\frac{1}{2}\right)=r(\cos\varphi+i\sin\varphi)$$

$$r=\frac{\sqrt{2}}{2}$$

$$\varphi=\operatorname{atan}-1=\cancel{-\frac{\pi}{4}+k\pi}=\cancel{\frac{3\pi}{4}+k\pi}=-\frac{\pi}{4}+2\pi=\frac{7\pi}{4}$$

$$\left(\frac{1}{2}-i\frac{1}{2}\right)^{18}=\left(\frac{\sqrt{2}}{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)\right)^{18}=\left(\frac{\sqrt{2}}{2}\right)^{18}\left(\cos\frac{18\cdot 7\pi}{4}+i\sin\frac{18\cdot 7\pi}{4}\right)=$$

$$=\left(\frac{\sqrt{2}}{2}\right)^{18}\left(\cos\frac{63\pi}{2}+i\sin\frac{63\pi}{2}\right)=\left(\frac{\sqrt{2}}{2}\right)^{18}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=\frac{1}{2^9}(0-i)=-\frac{i}{2^9}$$

$$\sqrt[4]{1+i}=\sqrt[4]{\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}=\sqrt[4]{\sqrt{2}}\left(\cos\frac{\frac{\pi}{4}+2k\pi}{4}+i\sin\frac{\frac{\pi}{4}+2k\pi}{4}\right);\ \underline{\underline{k=0,1,2,3}}$$

$$k=0;\ \sqrt[8]{2}\left(\cos\frac{\pi}{16}+i\sin\frac{\pi}{16}\right),\quad k=1;\ \sqrt[8]{2}\left(\cos\frac{\frac{\pi}{4}+2\pi}{4}+i\sin\frac{\frac{\pi}{4}+2\pi}{4}\right)=\sqrt[8]{2}\left(\cos\frac{9\pi}{16}+i\sin\frac{9\pi}{16}\right)$$

$$k=2;\ \sqrt[8]{2}\left(\cos\frac{\frac{\pi}{4}+4\pi}{4}+i\sin\frac{\frac{\pi}{4}+4\pi}{4}\right)=\sqrt[8]{2}\left(\cos\frac{17\pi}{16}+i\sin\frac{17\pi}{16}\right)$$

$$k=3;\ \sqrt[8]{2}\left(\cos\frac{\frac{\pi}{4}+6\pi}{4}+i\sin\frac{\frac{\pi}{4}+6\pi}{4}\right)=\sqrt[8]{2}\left(\cos\frac{25\pi}{16}+i\sin\frac{25\pi}{16}\right)$$

$$z_1=(1+i)$$
$$z_2=(1-i)$$
$$\Downarrow$$

CIKLOMETRIČNE ENAČBE

a) $z\cdot\bar{z}=-1 \Rightarrow (x+iy)(x-iy)=-1 \Rightarrow \boxed{\begin{aligned}x^2+y^2&=-1\\ x,y&\in\mathbb{R}\end{aligned}}$ | <u>ni rešitev v $\mathbb{R}$</u>

$z=x+iy$ R: $\{\ \}$

b) $2z^2-3\bar{z}^2=10i$

$$2(x+iy)^2-3(x-iy)^2=10i \Rightarrow 2x^2+4xiy-2y^2-3x^2+6xiy+3y^2=10i$$

$$-x^2+10xiy+y^2=10i+0 \qquad -x^2+y^2=0 \qquad -x^2+\frac{1}{x^2}=0 \qquad -x^4+1=0 \qquad x_1=-1$$

$$xy=1 \qquad -x^4+1=0 \qquad x^4=1 \qquad x_2=$$

$(x^2-1)(x^2+1)$

$(x+1)(x-1)(x^2+1)$

$x\in\mathbb{R}$

uporaba pol. zapisa

$z^3 = 1$ $\qquad z = \sqrt[3]{1(\cos 0^\circ + i\sin 0)} = 1\cdot\left(\cos\frac{0+2k\pi}{3} + i\sin\frac{0+2k\pi}{3}\right)$ $\quad k=0,1,2$

R: $k=0 \quad z_1 = 1$

$k=1 \quad z_2 = 1\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 1\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$k=2 \quad z_3 = 1\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = 1\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$-0{,}5$

$-\frac{1}{2} + \frac{\sqrt{3}}{2}$

~~$z\sqrt[4]{64+64\cdot 3} = \sqrt{256} = 16$~~

~~$r = 16$~~ $\qquad z = \sqrt[4]{-8+8i\sqrt{3}}$

$z^4 = -8+8i\sqrt{3}$

$\varphi = \text{atan} -\sqrt{3} = \frac{2\pi}{3}$ $\quad r = 16$ $\qquad$ ~~$z = \sqrt[4]{\;}$~~

$\sqrt[4]{16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)} = \cancel{2} \sqrt[4]{16}\left(\cos\frac{\frac{2\pi}{3}+2k\pi}{4} + i\sin\frac{\frac{2\pi}{3}+2k\pi}{4}\right)$ $k=0,1,$

$k=0 \quad \sqrt[4]{16}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \boxed{\sqrt{3}+i}$

$k=1$ $\boxed{-1+i\sqrt{3}}$ $\sqrt[4]{16}\left(\cos\frac{\frac{2\pi}{3}+2\pi}{4} + i\sin\frac{\frac{2\pi}{3}+2\pi}{4}\right) = \sqrt[4]{16}\left(\cos\frac{8}{12}\pi + i\sin\frac{8}{12}\pi\right) = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -1+i\sqrt{3}$ DN

$k=2$ $\boxed{-\sqrt{3}-i}$ $2\left(\cos\frac{\frac{2\pi}{3}+4\pi}{4} + i\sin\frac{\frac{2\pi}{3}+4\pi}{4}\right) = 2\left(\cos\frac{7}{6}\pi + i\sin\frac{7}{6}\pi\right) = 2\left(-\frac{\sqrt{3}}{2} + \left(-i\frac{1}{2}\right)\right) = -\sqrt{3} - i$

$k=3$

$\boxed{1-i\sqrt{3}}$ $2\left(\cos\frac{\frac{2\pi}{3}+6\pi}{4} + i\sin\frac{\frac{2\pi}{3}+6\pi}{4}\right) = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2\left(\frac{1}{2} + \left(-i\frac{\sqrt{3}}{2}\right)\right) = 1-i\sqrt{3}$

DN

$2z^3 = \left(-1 - i\sqrt{3}\right)^6$ $\qquad$ R: $z = 2\sqrt[3]{4}\left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right)$

$z^3 = \frac{1}{2}\left(-1-i\sqrt{3}\right)^6$ $\qquad k = 0,1,2$

$z = \sqrt[3]{\frac{1}{2}\left(-1-i\sqrt{3}\right)^6}$ ⟸ to ni navaden koren, je le oznaka za uporabo pri formuli.

$|z-2| = 3$

$|z+1| = 3$

$|x+iy-2| = 3$

$|x+iy+1| = 3$

$$\boxed{\begin{aligned} z_1 &= \frac{1}{2} + i\frac{3\sqrt{3}}{2} \\ z_2 &= \frac{1}{2} - i\frac{3\sqrt{3}}{2} \end{aligned}}$$

$z = x+iy = r(\cos\varphi + i\sin\varphi)$

$\sqrt{(x-2)^2+y^2} = 3 \;/^2$

$\sqrt{(x+1)^2+y^2} = 3 \;/^2$

$\left.\begin{aligned}(x-2)^2+y^2 &= 9 \\ (x+1)^2+y^2 &= 9\end{aligned}\right) -$

$(x-2)^2 = (x+1)^2 = 0$

$\cancel{x^2} - 4x + 4 = \cancel{x^2} + 2x + 1$

$-6x = 3$

$x = -\frac{3}{6} = -\frac{1}{2}$

$\left(\frac{3}{2}\right)^2 + y^2 = 9$

$y^2 = \frac{27}{4}$

$y = \pm\frac{3\sqrt{3}}{2}$

DN $z_1 \cdot \bar{z}_2 = \sqrt{2}$

$\frac{z_1}{\bar{z}_2} = i\sqrt{2}$

R: $z_1 = -1-i$

$z_2 = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$z_1 = 1+i$

$z_2 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$(2+i)z_1 + (2-i)z_2 = 6 \quad /(3+2i)$
$(3+2i)z_1 + (3-2i)z_2 = 8 \quad /(2+i)$

$\cancel{(2+i)(3+2i)z_1} + (2-i)(3+2i)z_2 = 6(3+2i)$
$\cancel{(3+2i)(2+i)z_1} + (3-2i)(2+i)z_2 = 8(2+i) \quad -$

$(2-i)(3+2i)z_2 - (2+i)(3-2i)z_2 = 18+12i-16+8i$

$z_2(\cancel{6}-3i+4i+\cancel{2}-\cancel{6}+i-\cancel{2}) = 2+4i$

$z_2(2i) = 2+4i$

$z_2 = \frac{2+4i}{2i} = \frac{2(1+2i)}{2i} = \frac{(1+2i)}{i} \Big/ \frac{i}{-i} = \underline{\underline{2-i}}$

$(2+i)z_1 + (2-i)(2-i) = 6$
$(2+i)z_1 + (4-2i-2i+i^2) = 6$
$z_1(2+i) + 3-4i = 6$

$z_1 = \frac{3+4i}{2+i} \cdot \frac{(2-i)}{(2-i)} = \frac{6-3i+8i-4i^2}{5} = \frac{10+5i}{5} = \boxed{2+i}$

---

podmnožice kompleksne ravnine

$\mathrm{Re}(z) + \mathrm{Im}(z^2) = 2$

$z = x+iy \Rightarrow z^2 = x^2 + 2ixy - y^2$

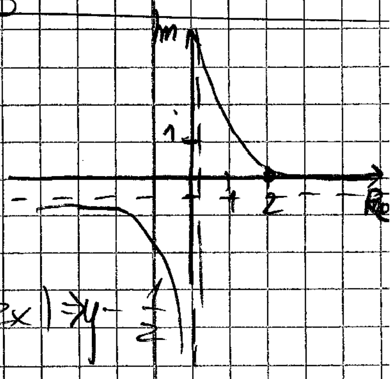
$x + 2xy = 2$ racionalna funkcija

$y = \frac{2-x}{2x}$

N: $2-x=0$
$x=2$

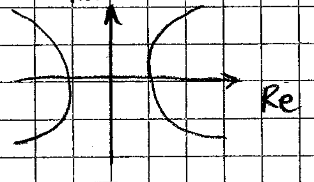
P: $2x=0$
$x=0$

A $(2-x) | (2x) \Rightarrow y = -\frac{1}{2}$



DN

$\mathrm{Re}(\bar{z}^2) = 4$



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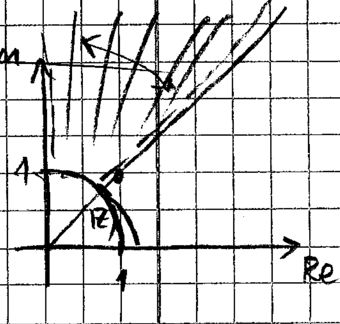
c) $\{z \in \mathbb{C};\ \frac{\pi}{4} < \underset{\varphi}{\underset{\Downarrow}{\arg}}(z) < \frac{\pi}{2} \text{ in } |z| > 1\}$



MAT REPETITORIJ

$$\left(\frac{-\sqrt{2}+i\sqrt{2}}{\sqrt{3}+i}\right)^6=\left(\frac{(-\sqrt{2}+i\sqrt{2})}{(\sqrt{3}+i)}\frac{(\sqrt{3}-i)}{(\sqrt{3}-i)}\right)^6=\left(\frac{-\sqrt{6}+i\sqrt{6}+i\sqrt{2}+\sqrt{2}}{3+1}\right)^6=\frac{(\sqrt{2}-\sqrt{6})+i(\sqrt{2}+\sqrt{6}}{4}$$

$$x=\frac{\sqrt{2}-\sqrt{6}}{4}\qquad r=\sqrt{x^2+y^2}=\sqrt{\frac{2-\cancel{4\sqrt{3}}+6+2+2+\cancel{4\sqrt{3}}+6}{4}}=\frac{4}{4}=\underline{1}$$

$$y=\frac{\sqrt{2}+\sqrt{6}}{4}\qquad \varphi=\operatorname{atg}\frac{y}{x}=\operatorname{atg}\frac{(\sqrt{2}+\sqrt{6})}{(\sqrt{2}-\sqrt{6})}=\operatorname{atg}\frac{(\sqrt{2}+\sqrt{6})^2}{(\sqrt{2}-\sqrt{6})(\sqrt{2}+\sqrt{6})}=$$

$$=\operatorname{atg}\frac{2+\sqrt{2}\cdot\sqrt{6}+6}{2-4}=\operatorname{atg}\frac{8+4\sqrt{3}}{-4}=\underline{-\operatorname{atg}(2+\sqrt{3})}$$

De Moivre

$$\boxed{\begin{aligned}&f \text{ liha}: f(-x)=-f(x)\\&f \text{ soda}: f(-x)=f(x)\end{aligned}}$$

$$(r(\cos\varphi+i\sin\varphi))^n=r^n(\cos(n\varphi)+i\sin(n\varphi))$$

$$1^6\cdot\left[\cos(-6\operatorname{atg}(2+\sqrt{3}))+i\sin(-6\operatorname{atg}(2+\sqrt{3}))\right]=-16$$

$$z^2=\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^3\qquad x=\frac{1}{2}\quad r=1\qquad z^2=1^3(\underbrace{\cos(-\pi)}_{-1}+i\underbrace{\sin(-\pi)}_{0})$$

$$y=-\frac{\sqrt{3}}{2}\quad \varphi=\operatorname{atg}(-\sqrt{3})=-\frac{\pi}{3}$$

1. CIKLOMETRIČNA ENAČBA 2. KOREN $z^2=-1$

$$z^n=w_0\qquad z=\sqrt[n]{w_0}$$

$$z_k=\sqrt[n]{r}\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right)\quad n=2,\ w_0=-1\qquad k=0,1,\ldots,(n-1)$$

$$z=|z|(\cos\alpha+i\sin\alpha)$$

a) $z^2=-1$

$$|z|^2(\cos 2\alpha+i\sin 2\alpha)=1\cdot(\cos\pi+i\sin\pi)$$

$$|z|^2=1\qquad \cos 2\alpha=\cos\pi$$

$$2\alpha=\pi+2k\pi$$

$$|z|=1$$

$$\alpha_k=\frac{\pi}{2}+k\pi;\ k=0,1$$

$$\alpha_0=\frac{\pi}{2},\ \alpha_1=\frac{3\pi}{2}$$

$$z_k=|z|(\cos\alpha_k+i\sin\alpha_k),\quad k=0,1$$

$$z_0=1\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=i$$

$$z_1=1\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=-i$$

$$z_1=\sqrt{1}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$$

b) $r=1$ $\quad z=\sqrt{-1}$

$\varphi=\pi$

$n=2$ $\quad z_0=\sqrt{1}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=i$

$$\begin{aligned}(3-i)z_1+(4+2i)z_2&=1+3i \quad /(4+2i)\\ (4+2i)z_1+(2-i)z_2&=3 \quad /(3-i)\end{aligned}$$

$$z_2=\frac{(1+3i)(4+2i)+3(3-i)}{(4+2i)^2-(2-i)(3-i)}=\frac{7+11i}{-7+21i}=\frac{280-70i}{490}=\frac{4-i}{7}=\frac{4}{7}-\frac{1}{7}$$

$$z_1=\frac{(1+3i)-(4+2i)z_2}{(3-i)}=\frac{(1+3i)-(4+2i)\left(\frac{4-i}{7}\right)}{(3-i)}=$$

$$z^2+(2i-3)z+5-i=0$$

$$z=x+iy$$

$$x^2+2ixy-y^2+(2i-3)(x+iy)+5-i=0$$

$$x^2+2ixy-y^2+2ix-3x-2y-3iy+5-i=0$$

$$x^2-y^2-3x-2y+5=0$$

$$2xy+2x-3y-1=0 \Rightarrow (2x-3)y=1-2x$$

$$x^2-\left(\frac{1-2x}{2x-3}\right)^2-3x-2\left(\frac{1-2x}{2x-3}\right)+5=0 \qquad y=\frac{1-2x}{2x-3}$$

$$x^2-\frac{1-4x+4x^2}{(2x-3)^2}-3x-\frac{2-4x}{2x-3}+5 \quad /(2x-3)^2 \qquad x\neq\frac{3}{2}$$

$$x^2(4x^2-12x+9)-4x^2+4x-1-(2-4x)(2x-3)-3x\,(2x-3)^2+5\,(2x-3)^2=0$$

$$4x^4-24x^3+69x^2-101x+52=0$$

$$\{\pm1,\pm2,\pm4,\pm13,\pm26,\pm52\}$$

$$\left\{\pm\frac{1}{4},\pm\frac{1}{2},\pm\frac{13}{2},\pm\frac{13}{4}\right\}$$

	4	-24	69	-101	52
1		4	-20	+49	
	4	-20	+49	-52	0

$$x_1=1 \qquad \boxed{z_1=1+i}$$
$$y_1=1$$

MAT - VAJE 4. 11. 2010

ZAPOREDJA

$f: \mathbb{N} \to \mathbb{R}$ ⇐ funkcija, ki naravna števila preslika v realna števila

$1 \mapsto a_1$

$2 \mapsto a_2$

$a_1, a_2, a_3, \ldots$

Monotonost:

- narašča: $a_{\boxed{n+1}} \geq a_{\boxed{n}}$
- strogo naraščujoče: $a_{n+1} > a_n$
- pada: $a_{n+1} \leq a_n$
- strogo pada: $a_{n+1} < a_n$

---

narašča $a_{n+1} > a_n$

⇓

2 kriterija monotonosti:

- $a_{n+1} - a_n \geq 0$ narašča

$\frac{a_{n+1}}{a_n} \geq 1$ narašča (! za zaporedja s pozitivnimi členi)

---

minimum

$\boxed{\min_n a_n}$ ... najmanjši člen zaporedja

$\boxed{\max_n a_n}$ ... največji člen zaporedja

$a_n = \frac{1}{n}$ $\min_n a_n \nexists$ → minimum takega zaporedja ne obstaja

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$\boxed{\inf_n a_n}$ ... infimum ali natančna/ največja spodnja meja

$\boxed{\sup_n a_n}$ ... supremum ali natančna/ najmanjša ~~spodnja~~ zgornja meja

obstajata vedno (lahko sta ∞ ali $-\infty$)

nista nujno člena zaporedja

Zapiši splošni člen zaporedja:

a) 2, 3, 4, 5, 6

$\boxed{a_n = n+1 \quad n=1,\ a_1 = 2}$

b) $1, -4, 9, -16, 25, -36, \dots$ alternira

$a_n = (-1)^{n+1} n^2$

$\boxed{(-1)^n, (-1)^{n+1}}$ → vsako alt. zaporedje ima ta člen

② Kdaj zaporedje narašča in kdaj pada?

a) $a_n = \frac{2^n}{n!}$

$2, 2, \frac{8}{6}, \frac{16}{24}, \dots$

Def:

$n \in \mathbb{N}$ $n! = n\cdot(n-1) : 2 \cdot 1$

$0! = 1$

$a_{n+1} \geq a_n$; narašča

- $a_{n+1} - a_n \geq 0$
- $\frac{a_{n+1}}{a_n} \geq 1$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{n!}}{\frac{2^n}{n!}} = \frac{2^{n+1}\cdot n!}{2^n(n+1)!} = \frac{\cancel{2^n}2\cdot n!}{\cancel{2^n}(n+1)!} = \frac{2\cdot \cancel{n!}}{(n+1)\cdot \cancel{n!}} = \boxed{\frac{2}{n+1} \leq 1}$$

$(n+1)\cdot \underbrace{n(n-1)(n-2)\cdots 2\cdot 1}_{n!}$

⇓ povsod pada

b) $a^n = e^{-n}$

strogo padajoče

c) $a_n = n^2 - 11n - 8$ $\qquad a_{n+1} - a_n \geq 0$

$(n+1)^2 - 11(n+1) - 8 = n^2 + 11n + 8 =$

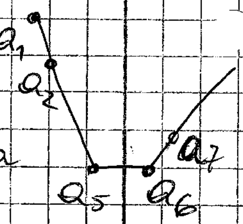
$= \cancel{n^2} + 2n + 1 - \cancel{11n} - 11 - \cancel{n^2} + \cancel{11n} =$

$2n - 10 = 0$

$2n = 10$

$\cancel{n = 5}$

$$\begin{cases} < 0 \ ; & n \leq 4 \text{ - pada} \\ = 0 \ ; & n = 5 \\ > 0 \ ; & n \geq 6 \text{ - narašča} \end{cases}$$



③ Določi največji in najmanjši člen, če obstajata, ter infimum in supremum zaporedja

a) $a_n = \frac{n}{n+1}$ $\qquad \frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}} = \frac{n+1}{n+2}\cdot\frac{n+1}{n} = \frac{(n+1)^2}{n(n+2)} = \frac{n^2+2n+1}{n^2+2n} > 1$

$a_1 = \frac{1}{2}$

$a_2 = \frac{2}{3}$

$a_3 = \frac{3}{4}$

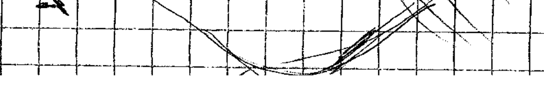
$\min_n a_n = a_1 = \frac{1}{2}$

$\max_n a_n$ = ne obstaja

zap. narašča proti 1, 1 ne doseže, pride le poljubno blizu

$\inf = \frac{1}{2}$

$\sup = 1$



b) $a_n = (n-1)\underline{\underline{(-1)}}^n$ → alternirajoče zaporedje

$a_1 = 0$
$a_2 = 1$
$a_3 = -2$
$a_4 = 3$
$a_5 = -4$

$\min\limits_n a_n \Longrightarrow$ ne obstaja
$\max\limits_n a_n \Longrightarrow$ ne obstaja
} števila grejo čez vso mejo

$\inf\limits_n a_n = -\infty$

$\sup\limits_n a_n = \infty$

c) DN $a_n = \left(1 - \frac{1}{n}\right)(-1)^n$

R: min in max ne obstajata

$\sup\limits_n a_n = 1$

$\inf\limits_n a_n = -1$

n! raste veliko hitreje kot potenca n

$a^n \ll n!$

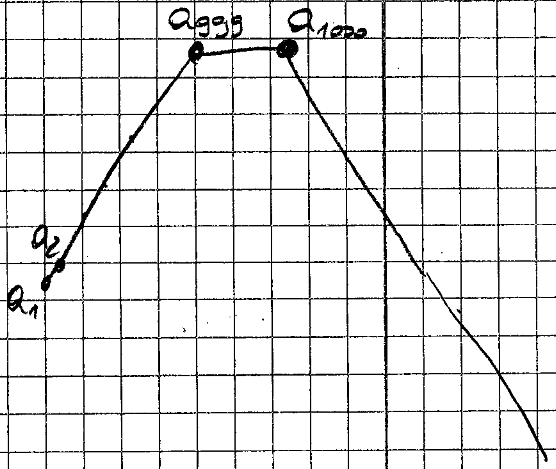
$a > 0, n \in \mathbb{N}$

d) $a_n = \frac{1000^n}{n!}$

$n \to \infty \Rightarrow 0$

$$\frac{a_{n-1}}{a_n} = \frac{\frac{1000^{n+1}}{(n+1)!}}{\frac{1000^n}{n!}} = \frac{1000^{n+1}\, n!}{1000^n\,(n+1)!} = \frac{\cancel{1000^n} \cdot 1000 \cdot n!}{\cancel{1000^n}\,(n+1)!} =$$

$$= \frac{1000}{n+1} \begin{cases} > 1 & n < 999 \\ = 1 & n = 999 \\ < 1 & n > 999 \end{cases}$$



$\min\limits_n a_n = \nexists$

$\max\limits_n a_n = a_{1000} = \frac{1000^{1000}}{1000!} = \frac{10^{3000}}{1000!}$

$\inf\limits_n a_n = 0$

$\sup\limits_n a_n = \max\limits_n a_n = \frac{10^{3000}}{1000!}$

e) $a_n = n^2 \left(\frac{9}{10}\right)^n$ DN

R: min $\nexists$
$\max\limits_n a_n = \sup\limits_n a_n = a_{19}$
$\inf\limits_n a_n = 0$

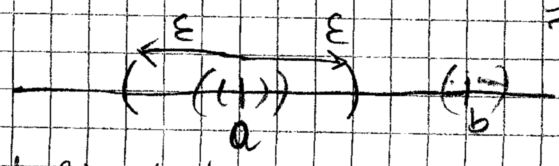
$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2\left(\frac{9}{10}\right)^{n+1}}{n^2\left(\frac{9}{10}\right)^n}$$

$$\frac{(n+1)^2 \cancel{\left(\frac{9}{10}\right)^n} \cdot \left(\frac{9}{10}\right)}{n^2 \cancel{\left(\frac{9}{10}\right)^n}} = \frac{9(n+1)^2}{10n^2}$$

$$\frac{9n^2 + 18n + 9}{10n^2}$$

Stekališče in limita

° stekališče



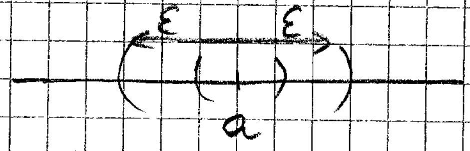
$f: \mathbb{N} \to \mathbb{R}$

poljubno majhno

ε-okolica $(\varepsilon \geqq 0)$

V vsaki ε-okolici točke a je neskončno členov zaporedja.

° limita



V vsaki ε-okolici točke a je neskončno členov zap, zunaj te ~~točke~~ okolice pa je le končno mnogo členov zap.

a limita $\Longrightarrow$ a stekališče

- a stekališče $\Longrightarrow$ a limita ~~nek~~ nekega podzaporedja

kriterij konvergence (= ima limito)

- a_n narašča in je navzgor omejeno $\Longrightarrow$ a_n konvergentno
- a_n pada in je navzdol omejeno $\Longrightarrow$ —||—
- a_n omejeno in ima eno samo stekališče $\Longrightarrow$ —||—

① Določi stekališča zaporedij. Ali so zaporedja konvergentna

a) $a_n = 1 - \frac{1}{n}$

R: eno stekališče: 1, konvergira

b) $a_n = \left(1 - \frac{1}{n}\right)(-1)^n$

$0, \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}$ $\Longrightarrow$ dve podzaporedji

$\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \dots \to 1$

$0, -\frac{2}{3}, -\frac{4}{5}, \dots \to -1$

2 stekališči; divergira

c) $0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{4}, \frac{3}{4}, 0, \frac{1}{8}, \frac{7}{8}, 0, \frac{1}{16}, \frac{15}{16}, 0, \dots$

Podzaporedja:

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots \to 0$

$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \dots \to 1$

$0, 0, 0, 0, \dots \to 0$

stekališči: 0, 1; divergentno

d) $a_n = \left(2 + \frac{3}{n}\right) \cos\frac{n\pi}{2}$

R: stekališča $-2, 0, 2$; divergentno

e) $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \to 0$

$\frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \frac{2}{6} \dots \to 0$

$\frac{3}{4}, \frac{3}{5}, \frac{3}{6}, \frac{3}{7} \dots \to 0$

$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \dots \to \frac{1}{2}$

$\boxed{n, m \in \mathbb{N},\ \frac{n}{m};\ n<m}$

ker jih je neskončno mnogo.

neskončno mnogo stekališč

② Izračunajte naslednje limite

a) $\lim\limits_{n\to\infty} \frac{8n^2+9n-6}{2n^3+3n+1} \Big/ :n^3 \lim\limits_{n\to\infty} \frac{\cancel{\frac{2}{n}}^{\to 0} + \cancel{\frac{9}{n^2}}^{\to 0} - \cancel{\frac{6}{n^3}}^{\to 0}}{2 + \cancel{3n^2}^{\to 0} + \cancel{n^3}^{\to 0}} = \frac{0}{2} = \underline{\underline{0}}$

$\boxed{\text{limita} \lim\limits_{n\to\infty} \frac{1}{n^k} = 0 \quad k \in \mathbb{N}}$

b) $\lim\limits_{n\to\infty} \frac{\sqrt{n^2+1}}{1+2n} = \lim\limits_{n\to\infty} \frac{\sqrt{\frac{n^2}{n^2} + \cancel{\frac{1}{n^2}}^{\to 0}}}{\cancel{\frac{1}{n}}^{\to 0} + 2} = \underline{\underline{\frac{1}{2}}}$

c) $\lim\limits_{n\to\infty} \underbrace{\left(\sqrt{n+1} - \sqrt{n}\right)}_{(a-b)} \left(\cdot \sqrt{n-1}\right) \cdot \underbrace{\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}}_{(a+b)} =$

$\lim\limits_{n\to\infty} = \frac{(n+1-n)\left(\sqrt{n-1}\right)}{\sqrt{n+1} + \sqrt{n}} = \lim\limits_{n\to\infty} \frac{\sqrt{n-1}}{\sqrt{n+1} + \sqrt{n}} \Big/ \sqrt{n}$

$\lim\limits_{n\to\infty} \frac{\sqrt{\frac{n}{n} - \cancel{\frac{1}{n}}^{\to 0}}}{\sqrt{\frac{n}{n} + \cancel{\frac{1}{n}}} + 1} = \cancel{b}\ \frac{1}{2}$

REPETITORIJ

$$a=\lim_{n\to\infty}\frac{n^2-1}{-n^2+2n}=\lim_{n\to\infty}\frac{1+\cancel{\frac{1}{n^2}}^{\to 0}}{-1+\cancel{\frac{2}{n}}^{\to 0}}=-1$$

$$b=\lim_{n\to\infty}\left(\sqrt{n^2-n}-\sqrt{n^2+n}\right)\frac{\left(\sqrt{n^2-n}+\sqrt{n^2+n}\right)}{\left(\sqrt{n^2-n}+\sqrt{n^2+n}\right)}=$$

$$\lim_{n\to\infty}\frac{\cancel{n^2}-n-\cancel{n^2}-n}{\sqrt{n^2-n}+\sqrt{n^2+n}}=\lim_{n\to\infty}\frac{-2n}{\sqrt{n^2-n}+\sqrt{n^2+n}}\Big/\frac{n}{n}=\lim_{n\to\infty}\frac{-2}{\sqrt{1-\cancel{\frac{1}{n}}^{0}}+\sqrt{1+\cancel{\frac{1}{n}}^{0}}}$$

$$\lim_{n\to\infty}\frac{-2}{2}=\underline{\underline{-1}}$$

$z=a+ib$ $\qquad r=\sqrt{2}$

$z=-1+i$ $\qquad \varphi=\operatorname{atg} 1=\frac{\pi}{4}+\pi=\frac{5\pi}{4}$

$$=\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$$

$$\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\dots+\frac{n-1}{n}\right)=\lim_{n\to\infty}\frac{\frac{(n-1)\,n}{2}}{n^2}=\lim_{n\to\infty}\frac{n-1}{2n}=\frac{1}{2}$$

$1+2+3+\dots+n-1$

$a_1=1$
$d=1$ $\qquad S_n=n\cdot a_1+\frac{n(n-1)}{2}\cdot d=\frac{(n-1)n}{2}$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e}$$

③ a) $\lim_{n\to\infty}\left(\frac{n^2+5}{n^2+3}\right)^{n^2+7}$

$$\lim_{n\to\infty}\left(\frac{n^2+3+2}{n^2+3}\right)^{n^2+7}=\lim_{n\to\infty}\left(1+\frac{2}{n^2+3}\right)^{n^2+7}=$$

$$\lim_{n\to\infty}\left(1+\frac{1}{\frac{n^2+3}{2}}\right)^{\left(n^2+7\right)\left(\frac{n^2+3}{2}\right)\left(\frac{2}{n^2+3}\right)}=e^{\lim\limits_{n\to\infty}\frac{2n^2+14}{n^2+3}}=\underline{\underline{e^2}}$$

b) $\lim_{n\to\infty}\left(1+\frac{1}{\frac{n^2+3}{2}}\right)^{\left(n^2+7\right)\left(\frac{n^2+3}{2}\right)\left(\frac{2}{n^2+3}\right)}$

$$m=\frac{n^2+3}{2}\qquad =\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{2m+4}=\lim_{m\to\infty}\left[\left(\left(1+\frac{1}{m}\right)^m\right)^2\cdot\left(1+\frac{1}{m}\right)^4\right]=$$

$n^2=2m-3$

$$=\left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^2\cdot\cancel{\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^4}^{\to 0}=\underline{\underline{e^2}}$$

④ $a_n = (-1)^n\left(\sqrt{n+3}-\sqrt{n}\right)$

$$\lim_{n\to\infty}\left(\sqrt{n+3}-\sqrt{n}\right) = \lim_{n\to\infty}\frac{\cancel{n}+3-\cancel{n}}{\sqrt{n+3}+\sqrt{n}} = 0$$

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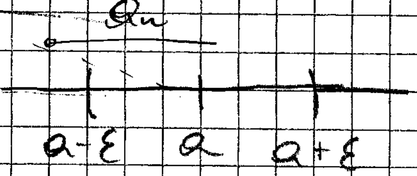
⑤ $a_n = \frac{3n^2+n-2}{n^2+2n+1}$ $\quad \varepsilon = \frac{1}{10}$



$$a = \lim_{n\to\infty} a_n = \underline{\underline{3}}$$

$$|a_n - a| > \varepsilon$$

$$\left|\frac{3n^2+n-2}{n^2+2n-1} - 3\right| > \frac{1}{10}$$

$$\left|\frac{3n^2+n-2 \quad -3n^2-6n-3}{n^2+2n+1}\right| > \frac{1}{10} \Rightarrow \left|\frac{-5n-5}{n^2+2n+1}\right| > \frac{1}{10}$$

$$\frac{5n+5}{(n+1)^2} > \frac{1}{10}$$

$$\frac{5\cancel{(n+1)}}{(n+1)^2} > \frac{1}{10}$$

$$50 > n+1$$

$$\boxed{n < 49}$$

komande za [illegible]

sqrt(2) = $\sqrt{2}$
exp(1) = e
2*sqrt(2) = $2\sqrt{2}$

$$a_n = \frac{2n-1}{3n+1} \qquad \frac{a_{n+1}}{a_n} = \frac{\frac{2n+1}{3n+4}}{\frac{2n-1}{3n+1}} = \frac{6n^2+5n+1}{6n^2+5n-4} > 1$$

$a_1 = \frac{1}{4}$

$a_2 = \frac{3}{7}$

$a_3 = \frac{1}{2}$

$a_4 = \frac{7}{13}$

$a_5 = \frac{9}{16}$

naraščajoče

$\min_n a_n = a_1 = \frac{1}{4} = \inf_n a_n$

$\max_n a_n =$ ne obstaja

$\lim_{n\to\infty} a_n = \frac{2}{3} = \sup_n a_n$

$$a_n = n^2\left(\frac{9}{10}\right)^n$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{n^2}\,\frac{\left(\frac{9}{10}\right)^{n+1}}{\left(\frac{9}{10}\right)^n} = \frac{9n^2+18n+9}{10n^2} > 1$$

$$9n^2+18n+9 > 10n^2$$

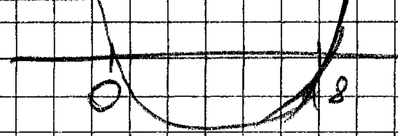
$$n^2-18n-9 < 0$$

$$n_{1,2} = \frac{18 \pm \sqrt{324+36}}{2} = 9 \pm 3\sqrt{10}$$

$n_1 \cong 0$

$n_2 \cong 18$

$n = 18$



$\max_n a_n = a_{19}$

$\inf_n a_n = 0$

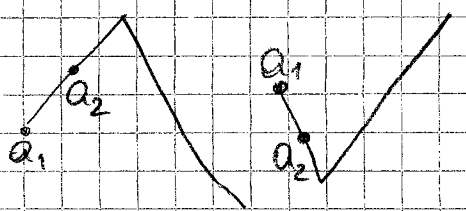
8.11.10

ZAPOREDJA

$f: \mathbb{N} \to \mathbb{R}$

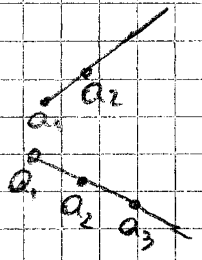
$1 \mapsto a_1$

$2 \mapsto a_2$



Monotonost zaporedij

- narašča $a_{n+1} \geq a_n$
- strogo narašča $a_{n+1} > a_n$
- pada $a_{n+1} \leq a_n$
- strogo pada $a_{n+1} < a_n$
- alternirajoča



$1, -1, 2, -2, 3, -3, \ldots$

$a_n = (-1)^n \cdot \frac{1}{n}$

Kriteriji monotonosti:

$a_{n+1} \geq a_n$ narašča

- $a_{n+1} - a_n \geq 0$ narašče
- $\frac{a_{n+1}}{a_n} \geq 1$ narašča $\Longleftarrow$ za pozitivne a_n!

$\max\limits_n a_n$ ... največji člen zaporedja } • ne obstajata vedno

$\min\limits_n a_n$ ... najmanjši člen zaporedja }

$\sup\limits_n a_n$ ... natančna / najmanjša zg. meja } • obstajata vedno $\Rightarrow$ $-\infty$, ∞

$\inf\limits_n a_n$ ... natančna / največja sp. meja } • nista nujno člena zap.

Če $\max\limits_n a_n$ obstaja, je enak $\sup\limits_n a_n$.

Če $\max\limits_n a_n$ ne obstaja, lahko vseeno najdemo $\sup\limits_n a_n$.

1. Zapiši splošni člen zaporedja:

a) $2, 3, 4, 5, 6, \ldots$

$a_n = n+1$

b) $1, -4, 9, -16, 25, -36, \ldots$ alternira

$a_n = (-1)^{n+1} n^2$

$$\boxed{\begin{array}{l} (-1)^n \rightarrow - + - + \\ (-1)^{n+1} \rightarrow + - + - \end{array}}$$

2.) Kdaj zaporedje narašča/pada:

a) $a_n = \frac{2^n}{n!}$ $\qquad \frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{\cancel{2^{n+1}}^{\,2}\cdot n!}{(n+1)!\cdot\cancel{2^n}}\ \frac{2\cancel{n!}}{(n+1)\cancel{n!}} = \frac{2}{(n+1)}$

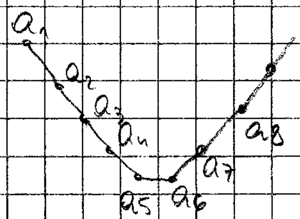
$\frac{2}{n+1} \leq 1$; pada povsod
(za vsak $n \in \mathbb{N}$)

b) $a_n = n^2 - 11n - 8$

$a_{n+1} - a_n = (n+1)^2 - 11(n+1) - 8 - n^2 + 11n + 8$
$\cancel{n^2} + 2n + 1 - \cancel{11n} - 11 - \cancel{8} - \cancel{n^2} + \cancel{11n} + \cancel{8}$

$2n - 10 = 0$
$n = 5$

$\begin{cases} < 0 ; n \leq 4 & \text{- pada} \\ = 0 ; \boxed{n = 5} & \leftarrow a_{n+1} - a_n = 0 \\ > 0 ; n \geq 6 & \text{- narašča} \end{cases}$



3.) Določi največji/najmanjši člen zap., če obstajata, ter sup. in inf.

a) $a_n = \frac{n}{n+1}$

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

- naraščanje/padanje

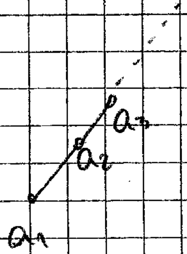
$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}} = \frac{(n+1)(n+1)}{n(n+2)} = \frac{n^2+2n+1}{n^2+2n} = 1 + \frac{1}{n^2+2n} > 1$ strogo narašča povsod

1

$\min\limits_n a_n = a_1 = \frac{1}{2} = \inf\limits_n a_n$

$\max\limits_n a_n$ ne obstaja

Zaporedje narašča proti 1,
se 1 poljubno približa, a
ga nikoli ne doseže



$\sup\limits_n a_n = 1$

b) $a_n = (n-1)(-1)^n$ $\qquad 0, 1, -2, 3, -4, \ldots$ alternirajoče

min in max ne obstajata, ker je zaporedje ↑↓ neomejeno

$\sup\limits_n a_n = \infty$

$\inf\limits_n a_n = -\infty$

c) $a_n = \frac{1000^n}{n!}$

$1000, 500\,000, \frac{10^6}{6}$

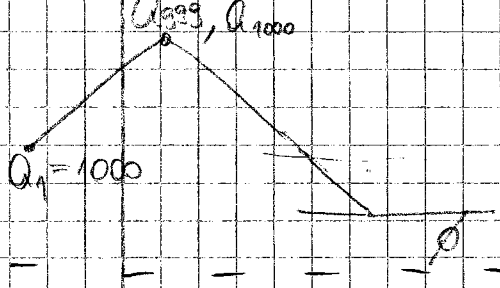
$\frac{a_{n+1}}{a_n} = \frac{1000^{n+1}}{(n+1)!} \cdot \frac{n!}{1000^n} \leq 10^3, \cdot 10^3$

$n! \gg a^n$

$a > 1$
$a \in \mathbb{R}$

$= \frac{n! \cdot 1000^{n+1}}{(n+1)! \cdot 1000^n} = \frac{1000}{(n+1)}$

$$\begin{cases} n < 1 ; & n < 999 \\ n = 0 , & n = 999 \\ n > 1 & n \geq 1000 \end{cases}$$



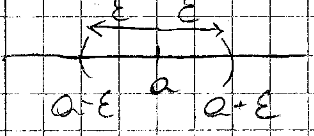
$n = 999$

$\sup_n a_n = \max_n a_n = a_{1000}$

$\inf_n a_n = 0$, min : ga ni

Stekališče in limita

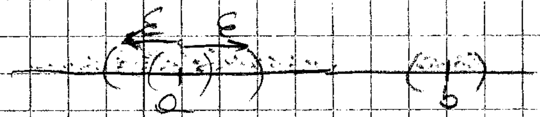
ε - okolica



$\varepsilon > 0$

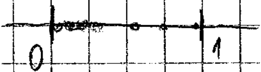
poljubno majhno število

stekališče

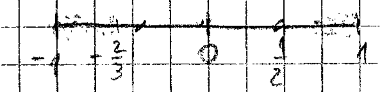


V vsaki ε-okolici št. a je neskončno mnogo členov zaporedja

$a_n = \frac{1}{n}$



$a_n = (-1)^n \left(1 - \frac{1}{n}\right)$



limita:

V vsaki ε-okolici št. a je neskončno mnogo členov zaporedja (stek.), zunaj nje pa je le končno mnogo členov.

Zaporedje, ki ima limito, je konvergentno, sicer je divergentno.

Kriteriji konvergence:

- narašča + navzgor omejeno
- pada + navzdol omejeno
- omejeno + 1 stekališče

} konvergentno

- ≥ 2 stekališči
- neomejeno

} divergentno

Vsako stekališče je limita nekega podzaporedja!

4.) Določi stekališča zaporedij; ali so zap. konvergentna ?

a) $a_n = \left(1-\frac{1}{n}\right)(-1)^n$

podzaporedja:

$0, \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}$

$0, \frac{1}{2}, \frac{3}{4}, \dots \longrightarrow 1$

$-\frac{2}{3}, -\frac{4}{5}, \dots \longrightarrow -1$

stekališči ste $-1, 1$

Zaporedje divergentno.

b) $0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{4}, \frac{3}{4}, 0, \frac{1}{8}, \frac{7}{8}, 0, \frac{1}{16}, \frac{15}{16}, 0 \dots$

$0, 0, 0, \dots \to 0$

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \to 0$

$\frac{1}{2}, \frac{3}{4}, \frac{7}{8} \longrightarrow 1$

stekališč 0, 1

divergentno

c) $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \dots$

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \dots \to 0$

neskončno mnogo zaporedij

$\frac{2}{3}, \frac{2}{4}, \frac{2}{5} \dots \to 0$

$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8} \dots \longrightarrow \frac{1}{2}$

$\frac{3}{4}, \frac{4}{5}, \frac{5}{6} \dots \to 0$

$\frac{1}{3}, \frac{2}{6}, \dots \longrightarrow \frac{1}{3}$

Vsa racionalna števila med 0 in 1.

- $\frac{n}{m}$; $n < m$; $n, m \in N$
- 0
- 1

} neskončno mnogo stekališč

Limita:

$\lim_{n\to\infty} C = C \quad \Rightarrow C, C, C, C, \dots \to C$

$\lim_{n\to\infty} \overset{a_n}{\boxed{\frac{1}{n^k}}} = 0$; $k \in N$

$\lim_{n\to\infty} \overset{a_n}{\boxed{n}} = \infty \quad \Rightarrow 1, 2, 3, \dots$ (divergentno!)

5.) Izračunaj naslednje limite

a) $\lim\limits_{n\to\infty} \frac{8n^2+3n-6}{2n^3+3n+1} \Big/ \frac{n^3}{n^3} = \lim\limits_{n\to\infty} \frac{\frac{8}{n}+\frac{3}{n^2}-\frac{6}{n^3}}{2+3/n^2+\frac{1}{n^3}} = 0$

b) $\lim\limits_{n\to\infty} \frac{\sqrt{n^2+1}}{1+2n} \Big/ \frac{\sqrt{n^2}}{\sqrt{n^2}} = \lim\limits_{n\to\infty} \frac{\sqrt{1+\cancel{\frac{1}{n^2}}^{\to 0}}}{\cancel{\frac{1}{n}}^{\to 0}+2} = \frac{1}{2}$

$\boxed{\text{npr: } \sqrt[3]{n^5-n^3+1} \rightsquigarrow n^{\frac{5}{3}}}$

$\boxed{\text{nedoločeni limitni izrazi: } \infty-\infty, \frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, 0^0, 1^\infty, \infty^0}$

c) $\lim\limits_{n\to\infty} \left(\sqrt{n+1}-\sqrt{n}\right)\cdot\sqrt{n-1} =$

$\lim\limits_{n\to\infty} \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\cdot\sqrt{n-1}\cdot\left(\sqrt{n+1}+\sqrt{n}\right)}{\left(\sqrt{n+1}+\sqrt{n}\right)} = \lim\limits_{n\to\infty} \frac{(n+1-n)\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n}} =$

$= \lim\limits_{n\to\infty} \frac{1\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n}} \Big/ \frac{\sqrt{n}}{\sqrt{n}} = \frac{\sqrt{1-\cancel{\frac{1}{n}}^{\to 0}}}{\sqrt{1+\cancel{\frac{1}{n}}^{\to 0}}+\sqrt{1}} = \frac{1}{2}$

d) $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)} \Big/ \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim\limits_{n\to\infty} \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}\,(n+1-n)} = \lim\limits_{n\to\infty} \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}} \Big/ \frac{\sqrt{n}}{\sqrt{n}}$

$\frac{\sqrt{1+\cancel{\frac{1}{n}}}+\sqrt{1}}{\sqrt{1}} = 2$

e) $\lim\limits_{n\to\infty} \frac{2^{n+1}+3^{n+1}}{2^n+3^n} \Big/ \frac{:3^n}{:3^n} = \lim\limits_{n\to\infty} \frac{\frac{2^n\cdot 2}{3^n}+3}{\frac{2^n}{3^n}+1} = \lim\limits_{n\to\infty} \frac{\cancel{\left(\frac{2}{3}\right)^n}^{\to 0}\cdot 2+3}{\left(\frac{2}{3}\right)^{n\to 0}+1} = 3$

Delimo z največjo osnovo, ker je kvocient manjši od 1.

f) $\lim\limits_{n\to\infty} \frac{2^{3n+1}-3^{2n+2}}{3^{2n-2}+1}$

$2^{3n}\cdot 2 = (2^3)^n\cdot 2 = 8^n\cdot 2$

$3^{2n+2} = 3^{2n}\cdot 9 = (3^2)^n\cdot 9 = 9^n\cdot 9$

$\lim\limits_{n\to\infty} = \frac{8^n\cdot 2-9^n\cdot 9}{9^n\cdot 3^{-2}+1} \Big/ \frac{:9^n}{:9^n} = \lim\limits_{n\to\infty} \frac{\cancel{\left(\frac{8}{9}\right)^n}^{\to 0}\cdot 2-9}{1\cdot\frac{1}{9}+\cancel{\left(\frac{1}{9^n}\right)}^{\to 0}} = \underline{-81}$

$\boxed{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = \lim\limits_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n}}$

11.11.2010

izračunaj limite zaporedij

a) $\lim\limits_{n\to\infty}\left(\frac{n+5}{n+3}\right)^{n} = \lim\limits_{n\to\infty}\left(\frac{(n+3)+2}{n+3}\right)^{n} = \lim\limits_{n\to\infty}\left(1+\frac{1}{\frac{n+3}{2}}\right)^{n\left(\frac{n+3}{2}\right)\left(\frac{2}{n+3}\right)} = \lim\limits_{n\to\infty} e^{\frac{2n}{n+3}} = e^{2}$

b) $\lim\limits_{n\to\infty}(n+3)(\ln(n+1)-\ln n) = \lim\limits_{n\to\infty}(n+3)\left(\ln\frac{n+1}{n}\right)$

$= \lim\limits_{n\to\infty}\ln\left(\frac{n+1}{n}\right)^{n+3} = \ln\lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^{n+3} =$

$= \ln\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+3} = \ln\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n\cdot\frac{1}{n}(n+3)} = \ln\lim\limits_{n\to\infty} e^{\frac{n+3}{n}} = \ln e =$

c) $\boxed{\text{DN: } \lim\limits_{n\to\infty}\left(\frac{2n^2+6}{2n^2+5}\right)^{4n^2+3}}$ $\boxed{R = e^{2}}$

d) $\lim\limits_{n\to\infty}\left(1+\frac{1}{\sqrt{n^2+n+1}}\right)^{1-\sqrt{n^2+n+1}} = \lim\limits_{n\to\infty}\underbrace{\left(1+\frac{1}{\sqrt{n^2+n+1}}\right)^{1}}_{1}\underbrace{\left(1+\frac{1}{\sqrt{n^2+n+1}}\right)^{-1}}_{e^{-1}}$

$= e^{-1}$

---

nedoločeni limitni izrazi

$\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, 0^{0}, \infty-\infty, 1^{\infty}, \infty$

Ugotovi, od katerega člena dalje se členi zaporedja $\{a_n\}$ razlikujejo od limite za $<\varepsilon$:

a) $a_n = \frac{n^2+n}{2n^2+1}, \varepsilon \frac{1}{10}$ $\quad |a_n - a| < \varepsilon$

$a = \lim\limits_{n\to\infty} a_n = \frac{n^2+n}{\lim\limits_{n\to\infty} 2n^2+1} = \frac{1}{2}$

$\left|\frac{n^2+n}{2n^2+1} - \frac{1}{2}\right| < \varepsilon$ $\quad \cancel{n_{1,2} = \frac{10 \pm \sqrt{100\varepsilon}}{}}$

$\cancel{\left|\frac{n^2+n-n^2-1}{2n^2+1}\right| < \varepsilon}$ $\quad \left|\frac{2n^2+2n-2n^2-1}{2(2n^2+1)}\right| < \frac{1}{10}$

$\cancel{\frac{n-1}{2n^2+1} < \frac{1}{10}}$ $\quad \frac{2n-1}{2(2n^2+1)} < \frac{1}{10}$

$\cancel{10n-10 < 2n^2+1}$ $\quad 20n-10 < 4n^2+2$

$\cancel{2n^2+11 < 10n \geq 0}$ $\quad 4n^2-20n+12 > 0$

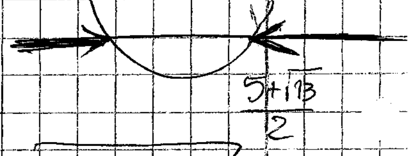
$n_{1,2} = \frac{5 \pm \sqrt{25-12}}{2}$

$n_{1,2} = \frac{5 \pm \sqrt{13}}{2}$

$n_1 = \frac{5+\sqrt{13}}{2}; n_2 = \frac{5-\sqrt{13}}{2}$

$n^2 - 5n + 3 > 0$

$\cancel{(n-2)(n-3) > 0}$



$\boxed{N \geq 5}$

2.) D.N.

$$a_n = \frac{n^2+2n}{n^2-2n+3} \; ; \; \varepsilon = \frac{1}{5}$$

$\boxed{R \quad N \geq 22}$

3.) $a_n = \dfrac{5^n-1}{5^n}$ $\qquad a = \lim\limits_{n\to\infty}\left(\dfrac{5^n-1}{5^n}\right) = \lim\limits_{n\to\infty}\left(1-\cancel{\dfrac{1}{5^n}}^{\to 0}\right) = 1$

$|a_n - a| < \varepsilon \qquad \varepsilon = 25^{-25}$

$$\left|\frac{5^n-1}{5^n} - 1\right| < \frac{1}{25^{25}} \Longrightarrow \left|\frac{\cancel{5^n}-1-\cancel{5^n}}{5^n}\right| < \frac{1}{25^{25}}$$

$$\frac{1}{5^n} < \frac{1}{25^{25}}$$

$$25^{25} < 5^n$$

$$(5^2)^{25} < 5^n$$

$$5^{50} < 5^n$$

$$n > 50$$

$$\boxed{N \geq 51}$$

4.) Dokaži, da je REKURZIVNO PODANO zaporedje konvergentno, in izračunaj limito:

konvergenca
1) narašča + navzgor omejeno
2) pada + navzdol omejeno
3) omejeno + 1 stekališče

$a_1 = 1; \; a_{n+1} = \frac{1}{5}a_n^2 + 1$

$n=1; \; a_2 = \frac{1}{5}a_1^2 + 1 = \frac{6}{5}$

$n=2: \; a_3 = \frac{1}{5}a_2^2 + 1 = \frac{1}{5}\left(\frac{6}{5}\right)^2 + 1 = \frac{36}{125} + 1 = \frac{161}{125}$

- zaporedje je navzgor omejeno s 2: $\boxed{a_n \leq 2}$

indukcija:

$\boxed{n=1}$ $\quad \underline{a_1 \leq 2}$

$\boxed{n \to n+1}$ $\quad$ i.p. $a_n \leq 2$

$a_{n+1} \leq 2$

$\frac{1}{5}a_n^2 + 1 \leq \frac{1}{5}\cdot 4 + 1$

$= \underline{\frac{9}{5} < 2} \; \checkmark$

- zaporedje narašča $\boxed{a_{n+1} \geq a_n}$

$\boxed{n=1}$ $\quad a_2 \geq a_1 \; \checkmark$

$\frac{6}{5} \geq 1$

$\boxed{n \to n+1}$ $\quad$ i.p. $a_{n+1} \geq a_n$

$a_{n+2} \geq a_{n+1}$

$a_{n+2} - a_{n+1} \geq 0$

$|a_{n+2}| - |a_{n+1}| =$

$$= \frac{1}{5} a_{n+1}^2 \cancel{+1} - \frac{1}{5} a_n^2 \cancel{-1} = \frac{1}{5}\left(a_{n+1}^2 - a_n^2\right) =$$

$$= \frac{1}{5}\underset{\geq 0}{\left(a_{n+1} + a_n\right)} \underbrace{\left(a_{n+1} - a_n\right)}_{\overset{\text{i.p.}}{\geq 0}} \geq 0 \checkmark$$

Zaporedje narašča in je navzgor omejeno.

$a = \lim\limits_{n\to\infty} a_n$

$a_{n+1} = \frac{1}{5} a_n^2 + 1$

$$\boxed{a = \frac{1}{5} a^2 + 1}$$

$a^2 - 5a + 5$

$a_{1,2} = \frac{5 \pm \sqrt{25-20}}{2}$

$a_{1,2} = \frac{5 \pm \sqrt{5}}{2}$

$\cancel{a_1 = \frac{5+\sqrt{5}}{2}}$ prevelika vrednost $\boxed{a_n \leq 2}$

$a_2 = \frac{5-\sqrt{5}}{2}$

$$\boxed{a = \frac{5-\sqrt{5}}{2}}$$

Ali je rekurzivno podano zaporedje konvergentno? Če je, izračunaj limito:

$a_1 = 1;$ $\boxed{a_{n+1} = 2(a_n + 1)}$ → vsak naslednji člen je vsaj 2-kratnik prejšnjega člena.

$a_2 = 2(a_1+1) = 4$

$a_3 = 2(a_2+1) = 10$

$a_4 = 2(a_3+1) = 22$

Zaporedje je neomejeno ⇒ divergentno. (nima limite)

ŠTEVILSKE VRSTE

$$a_1 + a_2 + a_3 + \ldots\ldots = \sum_{n=1}^{\infty} a_n$$

Vrsta konvergira, če konvergira zaporedje delnih vsot.

↑ vsota obstaja in je končna

$s_1 = a_1$
$s_2 = a_1 + a_2$
$s_3 = a_1 + a_2 + a_3$
$\vdots$

tedaj velja

$$\lim_{N\to\infty} s_N = \sum_{n=1}^{\infty} a_n$$

$s_N = a_1 + a_2 + \ldots + a_n$

REPETITORIJ

11. 11. 2010

$$\sum_{n=1}^{\infty} \frac{n+2}{n^3+3n^2+2n} \rightarrow a_n$$

PARCIALNI ULOMKI

$$a_n = \frac{n+2}{n^3+3n^2+2n} = \frac{n+2}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2} =$$

$$\frac{A(n+1)(n+2)+Bn(n+2)+Cn(n+1)}{n(n+1)(n+2)} = \frac{(A+B+C)n^2+(3A+2B+C)n+2A}{n(n+1)(n+2)} =$$

$A+B+C=0$

$3A+2B+C=1$

$2A=2 \Rightarrow \boxed{A=1}$

$B+C=-1$

$2B+C=-2$

$\boxed{B=-1}$

$\boxed{C=0}$

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

$$S_N = a_1 + a_2 + \ldots + a_N =$$

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{N} - \frac{1}{N+1}\right)$$

$$S_N = \left(1 - \frac{1}{N+1}\right)$$

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(1 - \overset{\to 0}{\cancel{\frac{1}{N+1}}}\right) = \underline{\underline{1}}$$

$a_1 = \sqrt{2}$

$a_{n+1} = \sqrt{2a_n}$

$a_2 = \sqrt{2a_1} = \sqrt{2\sqrt{2}} = 2^{\frac{3}{4}}$

$a_3 = \sqrt{2\cdot 2^{\frac{3}{4}}} = 2^{7/8}$

$a_4 = \sqrt{2\cdot 2^{7/8}} = 2^{15/16}$

1.) navzgor omejeno

$\underline{\underline{a_n < 2}}$ ✓

2.) naraščа

$a_{n+1} - a_n > 0$

$$a_{n+1} - a_n = \sqrt{2a_n} - a_n$$

$$= \frac{(\sqrt{2a_n} - a_n)(\sqrt{2a_n} + a_n)}{(\sqrt{2a_n} + a_n)} = $$

$$= \frac{2a_n + a_n^2}{\sqrt{2a_n} + a_n} = \frac{0 < a_n\,(\overbrace{2-a_n}^{2})}{\underbrace{\sqrt{2a_n}}_{0} + \underbrace{a_n}_{0}} > 0$$

navzgor omejena + naraščujoče = konv.

$$a_{n+1} = \sqrt{2a_n}$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n}$$

ker je funkcija zvezna, lahko limito nesemo pod koren

$$a = \sqrt{2a} \quad /^2$$

$a_1 = 2$

~~$a_2 = 0$~~

$$a^2 = 2a$$

$$a(a-2) = 0$$

$a_n = n^2 \left(\frac{15}{16}\right)^n$

$\max_n a_n = ?$

$a_{n+1} - a_n > 0$

$\frac{a_{n+1}}{a_n} > 1$

$$\frac{(n+1)^2 \cancel{\left(\frac{15}{16}\right)^{n+1}}}{n^2 \cancel{\left(\frac{15}{16}\right)^{n}}} > 1$$

$$\frac{15(n+1)^2}{16n^2} > 1$$

$$16n^2 < 15n^2 + 30 + 15$$

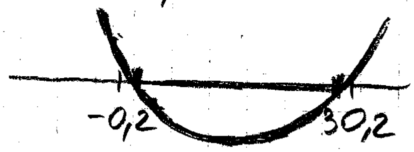
$$n^2 - 30n - 15 < 0$$

$$n_{1,2} = \frac{30 \pm \sqrt{900+60}}{2} = 15 \pm \sqrt{240} =$$

$$n_{1,2} = 15 \pm 4\sqrt{15}$$

$n_1 \doteq -0,2$

$n \doteq 30,2$



$n = 30$

$\underline{N \geq 31}$

$\sum_{n=1}^{\infty} \frac{n!}{n^n} \to a_n$

uporabimo kvocientni kriterij

$$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{(n+1)}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{\cancel{n!}\cancel{(n+1)}\, n^n}{\cancel{n!}(n+1)^n \cancel{(n+1)}} =$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \underline{\underline{e^{-1}}} \; \underline{\underline{< 1}}$$ konvergentno

$\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGENTNA

$\sum_{n=1}^{\infty} \frac{2n-1}{2^n}$ korenski kriterij

$\lim_{n\to\infty} \sqrt[n]{n} = 1$

$\lim_{n\to\infty} \sqrt[n]{c} = 1$; c = konst.

$$q = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{2n-1}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{2n-1}}{2} = \frac{1}{2} < 1$$ konvergira

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$5.146146146\ldots = 5 + \underline{0,146}146146\ldots$

$\frac{146}{1000} + \frac{146}{1000^2} + \frac{146}{1000^3} + \ldots$

$a_1 \quad a_2 \quad a_3$

$5 + \sum_{n=1}^{\infty} \frac{146}{1000^n}$ a_n

$\sum_{n=1}^{\infty} \underbrace{a q^{n-1}}_{a_n} = \frac{a}{1-q}$; $|q| < 1$

$$= 5 + \sum_{n=1}^{\infty} \frac{146}{1000^n} = 5 + \sum_{n=1}^{\infty} \left(\frac{146}{1000}\right)\left(\frac{1}{1000}\right)^{n-1} =$$

$a = a_1 = \frac{146}{1000}$

$q = \frac{a_2}{a_1} = \frac{1}{1000}$

$$= 5 + \frac{\frac{146}{1000}}{1 - \frac{1}{1000}} = 5 + \frac{146}{999} = \underline{\underline{\frac{5141}{999}}}$$

$\sum_{n=1}^{\infty} \frac{3}{\sqrt{n-1}}$ $\qquad n-1<n$ $\qquad \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \begin{cases} \alpha \le 1 \text{ DIV.} \\ \alpha > 1 \text{ KONV.} \end{cases}$

$$\sqrt{n-1} < \sqrt{n}$$

$$\frac{3}{\sqrt{n-1}} > \frac{3}{\sqrt{n}}$$

$$\sum_{n=2}^{\infty} \frac{3}{\sqrt{n-1}} > \sum_{n=2}^{\infty} \frac{3}{\sqrt{n}} = 3 \underbrace{\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}}_{\alpha<1}$$

DIV.

Če je neka vrsta > od divergentne vrste, je ta vrsta tudi DIV.

-||- > od konv., je lahko ta KONV. ali DIV

6) Seštej vrsto tako, da izračunaš limito zaporedja delnih vsot:

a) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \dots + \dots = \underline{\underline{\frac{1}{2}}}$

0

HARMONIČNA VRSTA

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots = \infty$$

PARCIALNI ALI DELNI ULOM~~K~~

$$\frac{1}{(4n^2-1)} = \frac{1}{(2n-1)(2n+1)} = \frac{1}{2n-1} + \frac{1}{2n+1} = \boxed{\frac{A}{2n-1} + \frac{B}{2n+1}}$$

$$= \frac{A(2n+1) + B(2n-1)}{(2n-1)(2n+1)}$$

$$1 = A(2n+1) + B(2n-1)$$

$$1 = 2An + A + 2Bn - B$$

$$\cancel{0n} + 1 = (2A + 2B)n + A - B \leftarrow$$

A, B konstanti n spremenljivke

$$n^1: 0 = 2A + 2B$$

$$n^0: 1 = A - B \quad / \cdot 2 \quad) -$$

$$-2 = 4B$$

$$B = \frac{-1}{2}$$

$$A = \frac{1}{2}$$

$$\frac{\frac{1}{2}}{2n-1} - \frac{\frac{1}{2}}{2n+1}$$

$$\lim_{N\to\infty} s_N = \lim_{N\to\infty} (a_1 + a_2 + \dots + a_n) =$$

$$= \lim_{N\to\infty} \Big(\underbrace{\frac{1}{2}\Big(\frac{1}{1} - \cancel{\frac{1}{3}}\Big)}_{n=1} + \frac{1}{2}\Big(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\Big) + \frac{1}{2}\Big(\cancel{\frac{1}{5}} - \frac{1}{7}\Big) + \dots + \frac{1}{2}\Big(\cancel{\frac{1}{2N-1}} - \frac{1}{2N+1}\Big)$$

$$\uparrow$$

$$a_n = \frac{\frac{1}{2}}{2n-1} - \frac{\frac{1}{2}}{2n+1} = \frac{1}{2}\Big(\frac{1}{2n-1} - \frac{1}{2n+1}\Big) ; \quad \underline{\lim_{N\to\infty}\Big(\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2N+1}\Big)} = \underline{\underline{\frac{1}{2}}}$$

DN

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$R = 1$

GEOMETRIJSKA VRSTA:

$$|q|<1$$

$$a + aq + aq^2 + \ldots = \sum_{n=0}^{\infty} aq^n = \sum_{n=1}^{\infty} aq^{n-1} = \boxed{\frac{a}{1-q}}$$

a = 1. člen vrste

q = kvocient dveh zap. členov (npr.: $\frac{a_2}{a_1}$)

Izračunaj vsoto geometrijske vrste:

a) $\sum_{n=2}^{\infty} \frac{11 \cdot 2^{2n}}{5^{n-3}}$

$$\begin{array}{r} 640 \\ +64 \\ \hline 704 \end{array}$$

$a = a_2 = \frac{11 \cdot 2^4}{5^{-1}} = 880$

$q = \frac{704}{880} = \frac{4}{5}$

$a_2 = a_3 = \frac{11 \cdot 2^6}{1} = 11 \cdot 64 = 704$

$$\begin{array}{r} 880 \cdot 5 \\ \hline 4400 \end{array}$$

$\sum_{n=2}^{\infty} = \frac{a}{1-q} = \frac{880}{\frac{1}{5}} = 5 \cdot 880 = \underline{4400}$

$a_n = n^2 \left(\frac{9}{10}\right)^n$

naraščanje / padanje

$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 \cancel{\left(\frac{9}{10}\right)^{n+1}}}{n^2 \cancel{\left(\frac{9}{10}\right)^{n}}} = \frac{(n+1)^2 \cdot 9}{10 \cdot n^2} =$ 15.11.2010

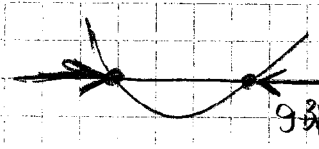
$= \frac{9}{10} \frac{n^2+2n+1}{n^2} < 1$

$9n^2 + 18n + 9 < 10n^2$

$\frac{18 \pm \sqrt{324+36}}{2} = \frac{18 \pm \sqrt{360}}{2}$

$$\begin{array}{r} 18 \cdot 18 \\ \hline 18 \\ 144 \\ \hline 324 \end{array}$$

$0 < n^2 - 18n - 9$



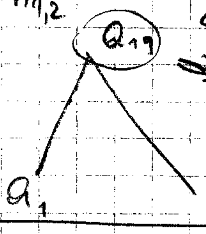
$9\,3\sqrt{10} = 18$

$n_{1,2} = \frac{18 \pm 6\sqrt{10}}{2} = \underline{\underline{9 \pm 3\sqrt{10}}}$

$n > 18$ ... pada

$n \leq 18$ ... narašča

$\underline{\underline{n \geq 19}}$



$\Rightarrow$ max a_n zap.

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① Izračunaj naslednje limite:

a) $\lim_{n\to\infty} \left(\frac{n+5}{n+3}\right)^n = \lim_{n\to\infty} \left(\frac{n+3+2}{n+3}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{\frac{n+3}{2}}\right)^n$

$\frac{n+3}{2} = m$

$2m - 3 = n$

$\lim_{n\to\infty} \left(1 + \frac{1}{\cancel{\frac{2m-3+3}{2}}}\right)^{2m-3} = \lim_{n\to\infty} \left(1 + \frac{1}{m}\right)^{2m-3} = e^{\lim_{n\to\infty} 2m-3} = e^2$

b) $\lim_{n\to\infty}(n+3)(\ln(n+1)-\ln n) = \lim_{n\to\infty}(n+3)\left(\ln\frac{n+1}{n}\right) =$

$\lim_{n\to\infty}\ln\left(\frac{n+1}{n}\right)^{n+3} = \lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^{\frac{n+3}{n}\cdot n} = \lim_{n\to\infty}\ln e^{\frac{n+3}{n}} = \underline{\underline{1}}$

c) $\boxed{\lim_{n\to\infty}\left(\frac{2n^2+6}{2n^2+5}\right)^{4n^2+3} \qquad R=e^2}$ DN

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d) ① $\lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n^2+n+1}}\right)^{(1-\sqrt{n^2+n+1})\frac{\sqrt{n^2+n+1}}{\sqrt{n^2+n+1}}} \lim_{n\to\infty} e^{\frac{1-\sqrt{n^2+n+1}}{\sqrt{n^2+n+1}}} = e^{-1}$

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② $\lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n^2+n+1}}\right)^{1-\sqrt{n^2+n+1}} = \lim_{n\to\infty}\left(1+\frac{1}{\sqrt{n^2+n+1}}\right)^{1}\cdot\left(1+\frac{1}{\sqrt{n^2+n+1}}\right)^{-\sqrt{n^2+n+1}} =$

$\underline{\underline{e^{-1}}}$

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② Ugotovi od katerega člena dalje se členi zaporedja $\{a_n\}$ ~~koždalja~~ razlikujejo od limite za $< \varepsilon$. ?

reši neenačbo:

$a = \lim_{n\to\infty}\frac{n^2+n}{2n^2+1} = \underline{\frac{1}{2}}$

a) $a_n = \frac{n^2+n}{2n^2-1}$; $\varepsilon = \frac{1}{10}$

$|a - a_n| < \varepsilon$

$\left|\frac{1}{2} - \frac{n^2+n}{2n^2+1}\right| < \frac{1}{10}$

$\left|\frac{\cancel{2n^2}+1-\cancel{2n^2}-2n}{2(2n^2+1)}\right| < \frac{1}{10}$

$\left|\frac{1-2n}{2(2n^2+1)}\right| < \frac{1}{10}$

$\wedge$

0

$-\left(\frac{1-2n}{2(2n^2+1)}\right) < \frac{1}{10}$

$\frac{2n-1}{2(2n^2+1)} < \frac{1}{10}$

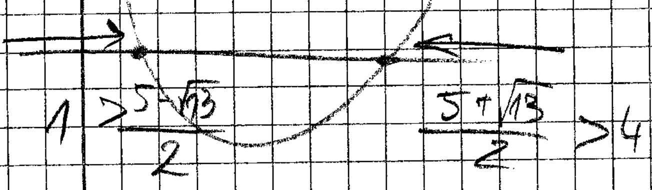
$5\,\cancel{10}(2n-1) < \cancel{2}(2n^2+1)$

$0 < 2n^2 - 10n + 6$

$n^2 - 5n + 3$

$n_{1,2} = \frac{5\pm\sqrt{25-12}}{2}$

$n_{1,2} = \frac{5\pm\sqrt{13}}{2}$



odgovor: $\boxed{N \geq 5}$

b) $a_n = \frac{n^2+2n}{n^2-2n+3}$; $\varepsilon = \frac{1}{5}$

R: 22 →

③ Koliko členov zap. s splošnim členom $a_n = \frac{5^n-1}{5^n}$ je od limite oddaljenih več kot 25^{-25}?

$|a_n - a| > 25^{-25}$ $\qquad \lim_{n\to\infty} a_n \left(1-\frac{1}{5^n}\right) = 1$

$$\left|\frac{5^n-1}{5^n} - 1\right| > \frac{1}{25^{25}}$$

$$\left|\frac{\cancel{5^n}-1-\cancel{5^n}}{5^n}\right| > \frac{1}{25^{25}}$$

$$\frac{1}{5^n} > \frac{1}{(5^2)^{25}} = \frac{1}{5^n} > \frac{1}{5^{50}}$$

$$5^n < 5^{50} \quad /\log$$

$$\underline{\underline{n < 50}}$$

$$N \le 49$$

④ Dokaži, da je rekurzivno podano zaporedje konvergentno in izračunaj limito:

$\boxed{a_1 = 1}$ $\quad a_{n+1} = \frac{1}{5} a_n^2 + 1$

naslednji člen izračunamo s pomočjo prejšnjega.

$$a_{n+1} = 3a_n - a_{n-1}^2 + 1 - 4a_{n-5}$$

$n=1$: $a_2 = \frac{1}{5} a_1^2 + 1 = \frac{6}{5}$

$n=2$: $a_3 = \frac{1}{5} a_2^2 + 1 = \frac{1}{5}\left(\frac{6}{5}\right)^2 + 1 = \frac{161}{125}$

$n=3$: $a_4 = \frac{1}{5} a_3^2 + 1 = \frac{1}{5}\left(\frac{161}{125}\right)^2 + 1 = \dots$

• narašča

konvergenca:
- narašča + navzgor ome
- pada + navzdol omejena
- omejeno + 1 stekališč

• zaporedje je navzgor omejeno: $a_n \le 2$

indukcija:

$\boxed{n=1}$: $a_1 \le 2$ ✓

$\boxed{n \to n+1}$: i.p. $a_n \le 2$ ✓

Ali velja tudi: $a_{n+1} \le 2$??

$$a_{n+1} = \frac{1}{5} \underset{\textcircled{2}}{a_n}^2 + 1 \le \frac{1}{5}\cdot 2^2 + 1 = \boxed{\frac{4}{5} + 1 = \frac{9}{5} < 2} \checkmark$$

Zaporedje narašča: $a_{n+1} \ge a_n \iff a_{n+1} - a_n \ge 0$

indukcija:

$\boxed{n=1}$ $a_2 \ge a_1$

$\frac{6}{5} \ge 1$

$\boxed{n \to n+1}$ i.p. $a_{n+1} \ge a_n$ oz. $a_{n+1} - a_n \ge 0$ ✓

Ali velja tudi $\boxed{a_{n+2} - a_{n+1} \ge 0}$?

$$\boxed{a_{n+2} - a_{n+1}} = \left(\frac{1}{5} a_{n+1}^2 + 1\right) - \left(\frac{1}{5} a_n^2 + 1\right) = \underbrace{\frac{1}{5} a_{n+1}^2 - \frac{1}{5} a_n^2}_{} =$$

$= \frac{1}{5}(a_{n+1}^2 - a_n^2) = \cancel{0}\ \frac{1}{5}(\underset{\overset{\vee}{0}}{a_{n+1}} + a_n)\underbrace{(a_{n+1} - a_n)}_{\overset{\vee\!\shortparallel}{0}\ \text{z.p.}} \geq 0$

$a_{n+1} = \frac{1}{5}a_n^2 + 1 > 0$

Vsi členi so pozitivni.

Zaporedje narašča in je navzgor omejeno $\Rightarrow$ konvergira

$a = \lim\limits_{n\to\infty} a_n$

$\lim\limits_{n\to\infty}(a_{n+1}) = \lim\limits_{n\to\infty}\left(\frac{1}{5}a^2+1\right)$

$a = \frac{1}{5}a^2 + 1 \quad / \cdot 5$

$a^2 - 5a + 5 = 0$

$a_{1,2} = \frac{5 \pm \sqrt{25-20}}{2}$

$a_{1,2} = \frac{5 \pm \sqrt{5}}{2}$

$\boxed{a_n \leq 2}$

$\frac{5+\sqrt{5}}{2} \doteq 3{,}6$

$\frac{5-\sqrt{5}}{2} \doteq 1{,}4$

Vsi členi so ≤ 2 zato je tudi limita ≤ 2

$\boxed{a = \frac{5-\sqrt{5}}{2}}$

---

⑤ Ali je rekurzivno podano zaporedje konvergentno, če je izračunaj limi

$a_1 = 1;\ a_n = 2(a_n + 1)$

$n=1;\quad a_2 = 2(1+1) = 4$

$n=2;\quad a_3 = 2(4+1) = 10$

$n=3;\quad a_3 = 2(10+1) = 22$

$\frac{10}{4} = \frac{2\cdot 5}{2\cdot 2} = 5/2$

$\frac{22}{10} = \frac{2\cdot 11}{2\cdot 5} = \frac{11}{5}$

$\boxed{\overset{a_1=1}{a_{n+1} > 2\cdot a_n}}$ $\Rightarrow$ zaporedje narašča čez vso mejo

- divergenca (neomejeno zap.)

$a_n > 2^n \Rightarrow$ neomejeno.

ŠTEVILSKE VRSTE

$a_1 + a_2 + a_3 + \ldots = \sum\limits_{n=1}^{\infty} a_n$

Vrsta konvergira (= vsota obstaja in je končno število), kadar konvergira zaporedje delnih vsot.

Zaporedje delnih vsot:

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

tedaj velja $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} s_n$

⑥ Seštej vrsto, tako da izračunaš limito zaporedja delnih vsot:

a) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \dots = \boxed{\frac{1}{2}}$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n} = \infty}$$

Harmonična vrsta

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(a_1 + a_2 + a_n + \dots + a_N\right) =$$

$$\lim_{N\to\infty} \left(\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \dots + \frac{1}{4N^2-1}\right)$$

Delni ali parcialni ulomki:

$$\frac{1}{4N^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{A}{(2n-1)} + \frac{B}{(2n+1)} = \frac{A(2n+1)+B(2n-1)}{(2n-1)(2n+1)}$$

Izenačimo števca:

$$1 = A(2n+1) + B(2n-1)$$

$$1 = 2An + A + 2Bn - B$$

$$1 = n(2A+2B) + A - B \;;\; n \Rightarrow \text{spremenljivka}$$

$A, B \Rightarrow$ konstanti (parametra)

$n^1: \; 0 = 2A + 2B$

$n^0: \; 1 = A - B \quad / \cdot 2$ ~~$A=B$~~ ~~$2B+2B=0$~~ ~~$4B=0$~~ ~~$1=1$~~

$$\left.\begin{array}{l} 0 = 2A+2B \\ 2 = 2A-2B \end{array}\right\} + \quad 2 = 4A \quad A = \frac{1}{2}, \; B = -\frac{1}{2}$$

$$= \lim_{N\to\infty} \left(\frac{1}{2}\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \frac{1}{2}\left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \dots + \frac{1}{2}\left(\cancel{\frac{1}{2N-1}} - \frac{1}{2N+1}\right)\right)$$

$$= \lim_{N\to\infty} \left(\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2n+1}\right) = \boxed{\frac{1}{2}}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \qquad R: 1}$$ DN

GEOMETRIJSKA VRSTA

$$a + aq + aq^2 + \dots = \sum_{n=0}^{\infty} aq^n = \sum_{n=1}^{\infty} aq^{n-1}$$

$$\boxed{\sum_{n=0}^{\infty} aq^n = \frac{a}{1-q} \;, \text{če } |q| < 1}$$

a = 1. člen vrste

q = kvocient dveh zaporednih členov = $\frac{2.\text{ člen}}{1.\text{ člen}}$

⑦ Izračunaj vsoto geometrijske vrste :

a) $\sum_{n=2}^{\infty} \frac{11\cdot 2^{2n}}{5^{n-3}} = \sum_{n=2}^{\infty} = \frac{11\cdot 4^{n}}{5^{n}\cdot 5^{-3}} = \sum_{n=2}^{\infty} \underbrace{11\cdot 5^{3}}_{a} \underbrace{\left(\frac{4}{5}\right)^{n}}_{q} = \boxed{\frac{a}{1-q}}$

$355\cdot 16$
55
330
880

(n=2) $a_1 = \frac{11\cdot 2^{4}}{5^{-1}} = 11\cdot 16\cdot 5 = \underline{\underline{880}}$ $n=3 = 11\cdot 2^{6}\cdot 1$

~~11~~

16^{2}

$q = \frac{\cancel{11}\cdot 2^{6}\cdot 1\cdot 2^{2}}{\cancel{11}\cdot 2^{4}\cdot 5} = \frac{4}{5} < 1$

$\frac{a}{1-q} = \frac{11\cdot 16\cdot 5}{1-\frac{4}{5}} = 11\cdot 16\cdot 5\cdot 5 = 4400$

80

400

4400

b) $\sum_{k=-2}^{\infty} \frac{2}{11\cdot 2^{2k}} = \frac{a}{1-q} = \frac{\frac{2^{5}}{11}}{1-\frac{1}{4}} = \frac{2^{5}4}{33} = \frac{2^{7}}{33} = \boxed{\frac{128}{33}}$

$(k=-2)\ a = \frac{2}{11\cdot 2^{(-2\cdot 2)}} = \frac{2}{11\cdot 2^{-4}} = \boxed{\frac{2^{5}}{11}\ \text{1. člen}}$ $\boxed{q = \frac{1}{4}}$

$(k=-1)\ a = \frac{2}{11\cdot 2^{-2}} = \boxed{\frac{2^{3}}{11}\ \text{2. člen}}$

c) $\sum_{n=3}^{\infty} \frac{1}{2\cdot 5^{n}}$ R: $\frac{1}{200}$

18.11.2010

$a_n = 6n^{2}\left(\frac{73}{81}\right)^{n}$

- naraščanje / padanje
  - $a_{n+1} - a_n$
  - $\frac{a_{n+1}}{a_n} = \frac{6(n+1)^{2}\left(\frac{73}{81}\right)^{n+1}}{6n^{2}\left(\frac{73}{81}\right)^{n}} = \frac{\cancel{6}(n+1)^{2}\cancel{\left(\frac{73}{81}\right)^{n}}\cdot\left(\frac{73}{81}\right)}{\cancel{6}n^{2}\cancel{\left(\frac{73}{81}\right)^{n}}} \quad \frac{73(n^{2}+2n+1)}{81n^{2}}$

$\left\{\begin{matrix} <1 \\ =1 \\ >1 \end{matrix}\right\}$ $\overset{?}{\geq} 1$ narašča

$73(n^{2}+2n+1) \geq 81n^{2}$

$146 \pm \sqrt{\quad}$

~~$73(n^{2}+2n+1)$~~ $=$ $0 \geq 8n^{2} - 146n - 73$

① Izračunaj vsoto geometrijske vrste

b.) $\sum_{n=3}^{\infty}\frac{11}{2\cdot5^n}=\frac{a}{1-q}=\frac{\frac{11}{2\cdot5^3}}{1-\frac{1}{5}}=\frac{11\cdot\cancel{5}}{4\cdot2\cdot5^{\cancel{3}2}}=\frac{11}{4\cdot50}=\frac{11}{200}$

$q=\frac{a_{n+1}}{a_n}=\frac{\frac{11}{2\cdot5^{n+1}}}{\frac{\cancel{11}}{2\cdot5^n}}=\frac{1}{5}=$ konst $\Rightarrow$ vrsta je geometrijska

$\boxed{q\in(-1,1)\text{ : konvergira}}$

1. člen vrste $=a=\frac{11}{2\cdot5^3}$

2. člen vrste $=a_2=\frac{11}{2\cdot5^4}$

$\Big\}\ q=\frac{\cancel{\frac{11}{2\cdot5^4}}}{\cancel{\frac{11}{2\cdot5^3}}}=\frac{1}{5}$

c) D.N. $\sum_{n=0}^{\infty}\frac{3^n}{5\cdot2^{2n}}$. R: $a=\frac{1}{5},\ q=\frac{3}{4},\ \frac{a}{1-q}=$

---

2) Zapiši decimalno število v obliki ulomka:

a) $x=5,\overline{146}$ $\qquad 5,146146,\ldots=5,\overline{146}$

$1000x=5146,\overline{146}$ $\quad -$

$999x=5141$

$x=\frac{5141}{999}$

2. način $\quad 5,146146\ldots=5+0,146+0,000146+\ldots$

$5+\frac{146}{1000}+\frac{146}{1000^2}+\frac{146}{1000^3}$

$5+\underbrace{\sum_{n=1}^{\infty}\frac{146}{1000^n}}_{\text{geo. vrsta}}$ $\qquad q=\frac{a_{n+1}}{a_n}=\frac{\cancel{\frac{146}{1000^n\cdot1000}}}{\cancel{\frac{146}{1000^n}}}=\frac{1}{1000}$

$\sum^{\infty} 5+\frac{\frac{146}{\cancel{1000}}}{\frac{999}{\cancel{1000}}}=5+\frac{146}{\cancel{1000}\,999}=\frac{5\cdot999+146}{999}=\frac{4995+14}{999}$

$\sum^{\infty}=\frac{5141}{999}$

b) $8,12381238123\ldots=8,1238$

$10\cdot0,\overline{8123}$ $\Leftarrow\!\!\Rightarrow$ $\qquad R=\frac{81230}{999}$

Kriteriji za konvergenco vrst s pozitivnimi členi

$\sum_{n=1}^{\infty} a_n;\ a_n > 0$ $\qquad \sum_{n=1}^{\infty}(-a_n) = -\sum_{n=1}^{\infty} a_n;\ a_n \geq 0$

Veljajo tudi za vrste z negativnimi členi

① kvocientni kriterij: $\qquad$ ② korensko koren:

$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ $\qquad q = \lim_{n\to\infty} \sqrt[n]{a_n}$

- če $q < 1$: konv.
- če $q > 1$: div.
- če $q = 1$: kriterij odpove ⟶ uporabimo drugega

ugotovi ali vrsta konvergira:

$\sum_{n=1}^{\infty} n = \infty$, div.

$\lim_{n\to\infty} a_n \neq 0 \Rightarrow$ div.

a) $\sum_{n=1}^{\infty} \frac{(3n)!}{(2n)!}$

$$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(3(n+1))!}{(2(n+1))!}}{\frac{(3n)!}{(2n)!}} =$$

$$\lim_{n\to\infty} \frac{\cancel{(2n)!}\,\cancel{(3n+3)}\,(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}{(2n+2)(2n+1)\cancel{(2n)!}\,\cancel{(3n)!}} = \frac{(3n+3)(3n+2)(3n+1)}{(2n+2)(2n+1)}$$

$$\lim_{n\to\infty}\left(\frac{27n^3+\dots}{4n^2+\dots}\right) = \infty$$ vrsta divergira

b) $\sum_{n=1}^{\infty}\left(\frac{1-n}{1+n}\right)^{n(n+1)}$; $q = \lim_{n\to\infty}\sqrt[n]{\left(\frac{n-1}{n+1}\right)^{n(n+1)}} = \lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^{n+1} = \lim_{n\to\infty}\left(\frac{n+1-2}{n+1}\right)^{n+1}$

$$\lim_{n\to\infty}\left(1-\frac{2}{n+1}\right)^{n+1} = \lim_{n\to\infty}\left(1-\frac{1}{\frac{n+1}{2}}\right)^{n+1} = \lim_{n\to\infty}\left(1-\frac{1}{\cancel{\frac{2m-1+1}{2}}}\right)^{2m-\cancel{1+1}}$$

$\left(\frac{n+1}{2}\right) = -\frac{m}{1}$

$-2m = n+1$

$n = 2m+1$

e^{-2}

c) $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

~~$q = 2^{n+1}(n+1)$~~

$$q = \lim_{n\to\infty} \frac{\frac{2^{n+1}(n+1)!}{(n+1)^{(n+1)}}}{\frac{2^n n!}{n^n}} = \lim_{n\to\infty} \frac{\cancel{2^n}\cdot 2\,\cancel{(n+1)}\,\cancel{n!}\; n^n}{(n+1)^n\cdot\cancel{n+1}\cdot\cancel{2^n}\,\cancel{n!}} = \lim_{n\to\infty}\frac{2n^n}{(n+1)^n} =$$

$$= \lim_{n\to\infty}\left(\frac{2n}{n+1}\right)^n = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} 2\left(\frac{n+1}{n}\right)^{-n} = 2e^{-1} = \frac{2}{e} < 1$$

konvergira

d) DN $\sum_{n=1}^{\infty} \frac{3^{2n+1}}{n\cdot 5^n}$ R: divergira

$3^{2n}\cdot 3 = 9^n\cdot 3$

VELJA

$\lim_{n\to\infty}\sqrt[n]{n} = 1$

$\lim_{n\to\infty}\sqrt[n]{c} = 1$; c=konst

e) $\sum_{n=1}^{\infty} \frac{2n\cdot n!}{(n+1)!} = \sum_{n=1}^{\infty} \frac{2n\,\cancel{n!}}{(n+1)\cancel{n!}} = \sum_{n=1}^{\infty} \frac{2n}{n+1}$

$a_n = \frac{2n}{n+1}$

$\lim_{n\to\infty} a_n = 2 \neq 0 \Rightarrow$ vrsta divergira

f) DN $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ $(n!)^2 = n!\cdot n!$ R: konvergira

3. Primerjalni kriterij

$0 \le a_n \le b_n$ od nekje naprej

potem:

- $\sum_{n=1}^{\infty} b_n$ konvergira ! $\Rightarrow \sum_{n=1}^{\infty} a_n$ konvergira

  MAJORANTA
- $\sum_{n=1}^{\infty} a_n$ divergira ! $\Longrightarrow \sum_{n=1}^{\infty} b_n$ divergira

  MINORANTA

MORAMO POZNATI, NI DANA

④ S pomočjo minorante ali majorante ugotovi konvergenco/divergenco vrste:

$$\boxed{\begin{array}{l} \sum_{n=1}^{\infty}\frac{1}{n} \Rightarrow \text{harmonična vrsta} \Rightarrow \text{divergira} \\ \sum_{n=1}^{\infty}\frac{1}{n^{\alpha}} \quad \begin{array}{l} \alpha>1 : \text{konvergira} \\ \alpha<1 : \text{divergira} \end{array} \\ \alpha\in\mathbb{R} \end{array}}$$

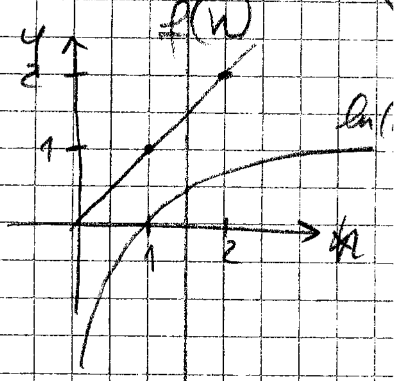
a) $\sum_{n=1}^{\infty}\frac{1}{\ln(n)}$ $\qquad \ln(n) < n \quad /(\;)^{-1}$

$\frac{1}{\ln(n)} = a_n$

$$\frac{1}{\ln(n)} > \frac{1}{n}$$

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{\ln(n)}} \Longleftarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n} \Rightarrow \text{divergira}}$$

divergira



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b) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n^2+1)}}$

$$\frac{1}{\sqrt{n(n^2+1)}} < \frac{1}{\sqrt{n(n^2)}} = \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$$

$$\sum_{n\to\infty}^{\infty}\frac{1}{n^{3/2}} \quad \boxed{\alpha = \frac{3}{2} > 1} \Longrightarrow \text{konvergira}$$

konv. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n^2+1)}}$

⑤ Ugotovi konvergenco vrste

$$\sum_{n=1}^{\infty}\frac{(2n+1)^3}{(n^3+1)^2}$$

$st(g) - st(p) = 6-3=3 \Longrightarrow$ konvergira

$$\frac{(2n+1)^3}{(n^2+1)^2} < \frac{(2n+1)^3}{n^4} \overset{\text{od nekje naprej}}{<} \frac{8n^4}{n^6} = \frac{8}{n^2}$$

$$\boxed{\begin{array}{l} \sum_{n=1}^{\infty}\frac{p(n)}{g(n)} \\ p, g \Rightarrow \text{polinoma} \end{array}}$$

$st(p) - st(g) \geq 2$ konvergira
$st(p) - st(g) \leq 1$ divergira

⑥ Konvergenca alternirajoče vrste

$$\pm\sum_{n=1}^{\infty}(-1)^n a_n, \; a_n > 0$$

$+ - + - \ldots$
$- + - + \ldots$

Leibnitzov kriterij: če zaporedje a_n od nekje naprej pada proti 0, alternirajoča vrsta konvergira.

• a_n padajoče

• $\lim\limits_{n\to\infty} a_n = 0$

$$\boxed{\begin{array}{l} \sum\limits_{n=1}^{\infty} \frac{1}{n} \text{ div} \\ \sum\limits_{n=1}^{\infty} (-1)^n \frac{1}{n} \text{ konv} \end{array}}$$

Ugotovi ali alternirajoča vrsta konvergira:

a) $\sum\limits_{n=1}^{\infty} (-1)^n \frac{1}{n}$

• $a_n = \frac{1}{n}$ pada? ✓

• $\lim\limits_{n\to\infty} a_n = 0$ ✓

} vrsta konvergira

b) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n \cdot n!}{n^n}$

$a_n = \frac{n!}{n^n}$

• padanje?

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{n^n (n+1)!}{(n+1)^{n+1} n!} =$$

$$= \frac{n^n \cancel{(n+1)}}{(n+1)^n \cancel{(n+1)}} = \left(\frac{n}{n+1}\right)^n < 1 \quad \text{pada ✓}$$

• $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n!}{n^n} = \lim\limits_{n\to\infty} \underset{\overset{\shortparallel}{1}}{\frac{n}{n}} \underset{\overset{\wedge}{1}}{\frac{(n-1)}{n}} \cdots \underset{\overset{\wedge}{1}}{\frac{2}{n}} \underset{\overset{\downarrow}{0}}{\frac{1}{n}} = 0$ ✓

$$\boxed{\text{c) } \sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 4^n}{n!} \qquad \text{DN} \qquad R = \text{konv.}}$$

Limitni izraz

$0 \cdot \infty \Rightarrow$ nedoločen

$0 \cdot \text{omejeno} = 0$

FUNKCIJE

$f: A \to B \;;\; A, B \subseteq \mathbb{R}$

$x \mapsto f(x)$

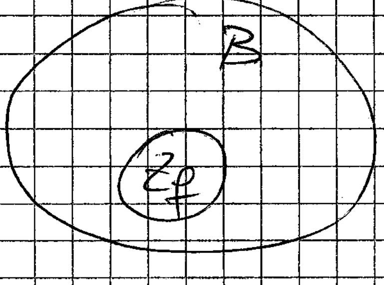
original slika

D_f; domena

kodomena $\boxed{\neq Z_f}$

$\boxed{Z_f \subseteq B}$

$\boxed{Z_f = \{f(x) \,; x \in A\}}$



Določi D_f območje naslednjih funkcij

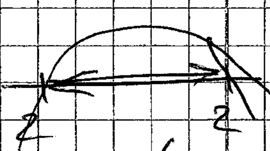
a) $f(x) = \ln 4-x^2 + \frac{1}{x^2-1}$ omejitve:

$D_f = \{x \in \mathbb{R}; 4-x^2>0$ in $x^2-1 \neq 0\}$ – ln : $4-x^2>0$

– —— : $x^2-1 \neq 0$

$4-x^2>0 \Rightarrow (2-x)(2+x)>0$

$x^2-1 \neq 0 \Rightarrow x \neq -1, 1$



$x \in (-2,2)$

$\boxed{D_f = \{ \cancel{x} (-2,2) \setminus \{-1,1\}}$

b) $f(x) = x + \sqrt{\frac{x-1}{x+1}}$

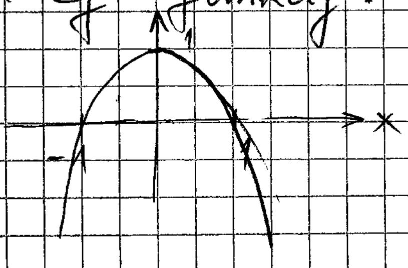
$R: (-\infty,-1) \cup [1,\infty)$

$\frac{(x-1)}{(x+1)} \geq \ / (x+1)^2 > 0$ DN

$x+1 \neq 0$

Določi D_f in Z_f funkcij:

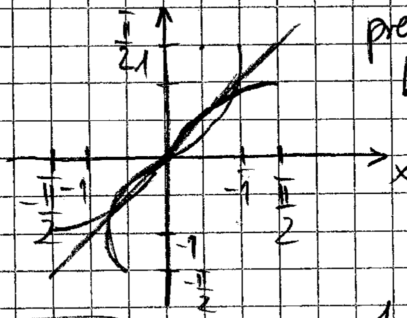
a) $f(x) = 1-x^2$



$D_f = \mathbb{R}$

$Z_f = (-\infty, 1]$

b) $\arcsin \frac{2x}{x+1}$



preslikamo preko simetrale lihih kvadrantov.

$D_f = [-1, 1]$

$Z_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$-1 \leq \frac{2x}{x+1} \leq 1$

$-1 \leq \frac{2x}{x+1} \leq 1$

– : $x+1 \neq 0 \Rightarrow x \neq -1$ *

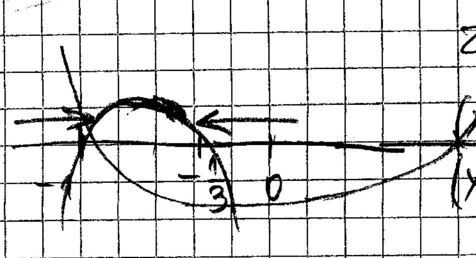
$-1 \leq \frac{2x}{x+1} / (x+1)^2 > 0$ in $\frac{2x}{x+1} \leq 1 / (x+1)^2$

$-(x+1)^2 \leq 2x(x+1)$

$(x+1)(-x-1-2x) \leq 0$

$(x+1)(-1-3x) \leq 0$

$x \in (-\infty,-1] \cup [-\frac{1}{3},\infty)$



$2x(x+1) \leq (x+1)^2$

$(x+1)(2x-x-1) \leq$

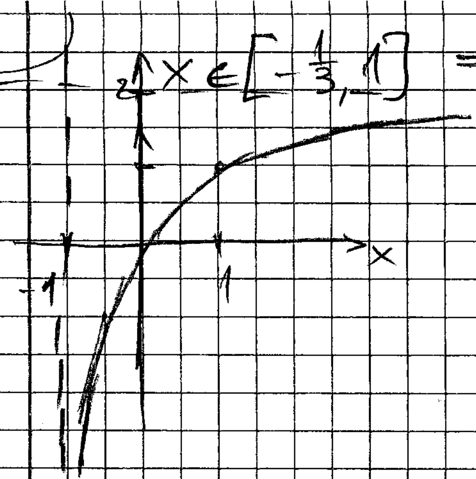
$(x+1)(x-1) \leq 0$

$x \in [-1,1]$

$x \in \{-1\} \cup [-\frac{1}{3}, 1]$

$\boxed{D_f : [-\frac{1}{3}, 1]}$

Z_f



$x \in \left[-\frac{1}{3}, 1\right] \Rightarrow \frac{2x}{x+1} \in [-1, 1] \Rightarrow$

$a\sin \frac{2x}{x+1} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$Z_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right.$

REPETITORIJ $\boxed{\text{Kol. 2001}}$ 18. 11. 2010

① zapiši kot interval ali kot unijo intervalov:

a) $\{x,\ x - \sqrt{3x} \geq 0\}$

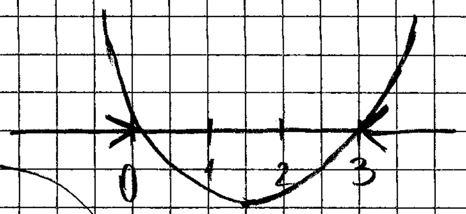
$x - \sqrt{3x} \geq 0$

$x \geq \sqrt{3x}$

$x^2 \geq 3x$

$x^2 - 3x \geq 0$

$x(x-3) \geq 0$



koren ne more biti negativen

$R: x \in (\cancel{-\infty}, 0] \cup [3, \infty)$

$R: x \in \{0\} \cup [3, \infty)$

b) $\{x,\ x \geq \sqrt{\ln(-\frac{1}{x})}\}$

$x \geq \sqrt{\ln(-\frac{1}{x})}\ /^2$

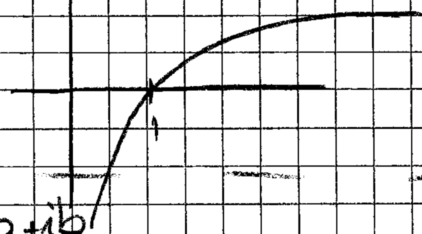
$x^2 \geq \ln(-\frac{1}{x})$

$-\frac{1}{x} > 0$

$x < 0$

$-\frac{1}{x} \geq 1$

$x \leq 1$



$R: \{\}$

② $|\bar{z} - 8| = 3iz$ $z = a + ib$

$|a - ib - 8| = 3i(a + ib)$

$\underbrace{\sqrt{(a-8)^2 + b^2}}_{Re} = \underbrace{3ai}_{Im} - \underbrace{3b}_{Re}$

$3a = 0$

$a = 0$

$\sqrt{(a-8)^2 + b^2} =$

$\sqrt{(a-8)^2 + b^2} = -3b$

$\sqrt{64 + b^2} = -3b\ /^2$

izraz pod korenom je > 0, zato je desna stran tudi > 0

$64 + b^2 = 9b^2$

$64 = 8b^2$

$b^2 = 8$

$\boxed{b_1 = -\sqrt{8}}$

$\cancel{b_2 = +\sqrt{8}}$

$z = -i\sqrt{8}$

③

	$a_n = \frac{(-1)^n}{n^2 - 4n + 10}$	$b_n = \frac{3n}{n+1}$	
največji člen	$\frac{1}{6} = a_2$	ne obstaja	
najmanjši člen	$-\frac{1}{7} = a_1 = a_3$	$\frac{3}{2}$	
natančna zg. meja	$\frac{1}{6} = a_2$	3	(približuje
limita	0	3	

$a_1 = -\frac{1}{7}$

$a_2 = \frac{1}{6}$

$a_3 = -\frac{1}{7}$

$a_4 = \frac{1}{10}$

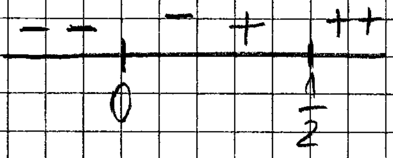
$a_5 = -\frac{1}{15}$

$$\frac{b_{n+1}}{b_n} = \frac{\frac{3(n+1)}{n+2}}{\frac{3n}{n+1}} = \frac{(3n+3)(n+1)}{(n+2)\,3n} = \frac{3n^2+6n+3}{3n^2+6n} > 1$$

KOL. 2003

① $\left|\frac{2x-1}{x}\right| < 2 \;/\; |x|$ $\qquad \left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

$|2x-1| < 2|x|$



1.) $x < 0$

$-2x+1 < -2x$

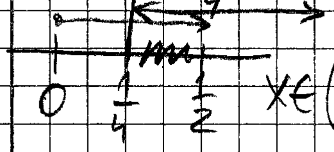
$1 < 0$ //

2.) $0 \leq x < \frac{1}{2}$

$-2x+1 < 2x$

$1 < 4x$

$x > \frac{1}{4}$



$x \in \left(\frac{1}{4}, \frac{1}{2}\right)$

3.) $x \geq \frac{1}{2}$

$2x-1 < 2x$

$-1 < 0$

$x \in \left[\frac{1}{2}, \infty\right)$

$$\boxed{R: \left(\frac{1}{4}, \infty\right)}$$

② $z^3 + 2 + 2i\sqrt{3} = 0$

$z^3 = -2 - 2i\sqrt{3}$

$\varphi = \operatorname{atg}\sqrt{3} = \frac{\pi}{3} + \pi = \frac{4\pi}{3}$

$r = 16$

$z_0 = \sqrt[3]{4}\left(\cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9}\right)$

$z_1 = \sqrt[3]{4}\left(\cos\frac{\frac{4\pi}{3}+2\pi}{3} + i\sin\frac{\frac{4\pi}{3}+2\pi}{3}\right) = \sqrt[3]{4}\left(\cos\frac{10\pi}{9} + i\sin\frac{10\pi}{9}\right)$

$z_2 = \sqrt[3]{4}\left(\cos\frac{16\pi}{9} + i\sin\frac{16\pi}{9}\right)$

③ $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\sqrt{n}\left(\sqrt{n+7}-\sqrt{n}\right)\left(\sqrt{n+7}+\sqrt{n}\right)}{\sqrt{n+7}+\sqrt{n}} = \lim_{n\to\infty}\frac{\sqrt{n}\left(\cancel{n}+7-\cancel{n}\right)}{\sqrt{n+7}+\sqrt{n}} = \frac{7}{2}$$

1) Zapiši decimalno število v obliki ulomka:

~~18~~.
22.11.201

a) ① $5{,}146146\ldots = 5{,}\overline{146}$

$x = 5{,}\overline{146}$

$1000x = 5146{,}\overline{146}$

$999x = 5141$

$x = \frac{5141}{999}$

② $5{,}146146\ldots = 5 + \frac{146}{1000} + \frac{146}{1000^2} + \frac{146}{1000^3} + \ldots$

$= 5 + \sum_{n=1}^{\infty} \frac{146}{1000^n}$ geometrijska vrsta $= 5 + \frac{146}{999} = \boxed{\frac{5141}{999}}$

$\frac{\overset{\to 1.\,člen}{a}}{1-q}$ $\quad q \to \frac{a_2}{a_1}$ $=$

$q = \frac{a_2}{a_1} = \frac{\frac{\cancel{146}}{\cancel{1000^2}}}{\frac{\cancel{146}}{\cancel{1000}}} = \frac{1}{1000}$

$\frac{\frac{146}{1000}}{1-\frac{1}{1000}} = \frac{\frac{146}{\cancel{1000}}}{\frac{999}{\cancel{1000}}} = \frac{146}{999}$

KRITERIJ ZA KONVERGENCO VRST S POZITIVNIMI ČLENI

$\sum_{n=1}^{\infty} a_n, \quad a_n > 0$

$-\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-a_n), \quad a_n > 0$

Vrste z negativnimi členi

1. Kvocientni kriterij

$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

2. Korenski kriterij

$q = \lim_{n\to\infty} \sqrt[n]{a_n}$

- $q < 1$; konvergira
- $q > 1$; divergira
- $q = 1$; kriterij odpove $\Rightarrow$ uporabimo kakšen drug kriterij

Ugotovi ali vrsta konvergira

a) $\sum_{n=1}^{\infty} \frac{(3n)!}{(2n)!} \to a_n$

$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(3(n+1))!}{(2(n+1))!}}{\frac{(3n)!}{(2n)!}} = \lim_{n\to\infty} \frac{\frac{(3n+3)!}{(2n+2)!}}{\frac{(3n)!}{(2n)!}} = \lim_{n\to\infty} \frac{\frac{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}}{\frac{\cancel{(3n)!}}{\cancel{(2n)!}}}$

$= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{27n^3 + \ldots}{4n^2 + \ldots} = \infty > 1$ - divergira

st. števca > st. imenovalca

b) $\sum_{n=1}^{\infty} \left(\frac{n-1}{n+1}\right)^{n(n+1)}$

$$q = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n-1}{n+1}\right)^{n(n+1)}} = \lim_{n\to\infty} \left(\frac{n-1}{n+1}\right)^{(n+1)} = \lim_{n\to\infty} \left(\frac{n-1}{n+1}\right)^{n} \cdot \left(\frac{n-1}{n+1}\right)^{1} =$$

$$= \lim_{n\to\infty} \left(\frac{n-2+1}{n+1}\right)^{n} \left(\frac{n-1}{n+1}\right)^{1} = \lim_{n\to\infty} \left(1 - \frac{2}{n+1}\right)^{n} \left(\frac{n-1}{n+1}\right)^{1} = \lim_{n\to\infty} \left(1 + \frac{1}{-\left(\frac{n+1}{2}\right)}\right)^{n} \left(\frac{n-1}{n+1}\right)^{1} =$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{-\left(\frac{2m \cancel{+n+1}}{2}\right)}\right)^{-2m-1} \left(\frac{n-1}{n+1}\right)^{1} = \underline{e^{-2}}$$

$-\left(\frac{n+1}{2}\right) = m$

$2m-1 = n$

$\frac{n-1}{2} = m$

$-n = 2m+1$

$n = -1-2m$

$-2m-1$

c) $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

$$q = \lim_{n\to\infty} \frac{\frac{2^{(n+1)} \cancel{(n+1)} \cancel{n!}}{(n+1)^n \cancel{(n+1)}}}{\frac{2^n \cancel{n!}}{n^n}} = \lim_{n\to\infty} \frac{\cancel{2^n} \cdot 2 \cdot n^n}{\cancel{2^n} (n+1)^n} = \lim_{n\to\infty} \left(\frac{2n}{(n+1)}\right)^n$$

$$2 \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = 2 \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n} = 2 \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-n} = \underline{\underline{\frac{2}{e}}}$$

d) $$\sum_{n=1}^{\infty} \frac{2n \cdot n!}{(n+1)!} = \sum_{n=1}^{\infty} \frac{2n \cancel{n!}}{(n+1)\cancel{n!}} = \sum_{n=1}^{\infty} \frac{2n}{n+1} = a_n$$

$\lim_{n\to\infty} a_n \neq 0 \implies$ vrsta divergira

vrsta konvergira $\implies \lim_{n\to\infty} a_n = 0$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{n+1} = 2 \neq 0$

divergira

PRIMERJALNI KRITERIJ

$0 \leq a_n \leq b_n$ od nekje naprej

- če $\underbrace{\sum_{n=1}^{\infty} b_n}_{\text{MAJORANTA}}$ konvergira $\implies \sum_{n=1}^{\infty} a_n$ konvergira
- če $\underbrace{\sum_{n=1}^{\infty} a_n}_{\text{MINORANTA}}$ divergira $\implies \sum_{n=1}^{\infty} b_n$ divergira

predznanje: referenčna vrsta; sami jo izberemo

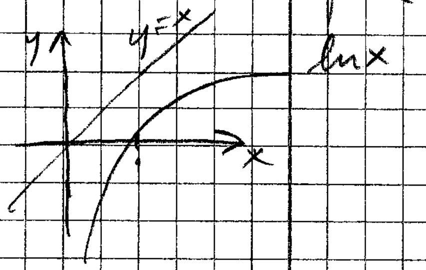
$\sum_{n=1}^{\infty} C^{\neq 0}$ divergira | $\sum_{n=1}^{\infty} n^k$ div. $k \in \mathbb{N}$ | $\sum_{n=1}^{\infty} \frac{1}{n}$ div. } harmonična vrsta | $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$, $\alpha \in \mathbb{R}$: $\alpha > 1$ konv, $\alpha \leq 1$ div.

$$\sum_{n=1}^{\infty} \frac{p(n)}{q(n)}$$ kon: $st(q)-st(p) \geq 2$

$p, q \rightarrow$ polinoma div: $st(q)-st(p) \leq 1$

S pomočjo minorante ali majorante ugotovi, ali vrsta konvergira

a) $\sum_{n=1}^{\infty} \frac{1}{\ln(n)}$ $\ln(n) < n$ $\boxed{\frac{1}{\ln(n)} > \frac{1}{n}}$



$\sum_{n=1}^{\infty} \frac{1}{n} \rightarrow$ divergira (harmonična)

MINORANTA

$\sum_{n=1}^{\infty} \frac{1}{\ln(n)} \rightarrow$ divergira

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n^2+1)}} \qquad \frac{1}{\sqrt{n(n^2+1)}} < \frac{1}{\sqrt{n \cdot n^2}} = \frac{1}{n^{3/2}}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n^2+1)}} \Longleftarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \quad \alpha = \frac{3}{2} > 1 \text{ konvergira}}$$

konv

Ugotovi konvergenco vrste

$$\sum_{n=1}^{\infty} \frac{(2n+1)^3}{(n^3+1)^2} \qquad \left.\begin{array}{l} st(p) = 3 \\ st(q) = 6 \end{array}\right\} \quad st(q)-st(p) = 6-3 = 3 \geq 2$$

vrsta konvergira

od nekje naprej je večje

$$\frac{(2n+1)^3}{(n^3+1)^2} < \frac{(2n+1)^3}{n^6} < \frac{n^4}{n^6} = \frac{1}{n^2} \qquad \alpha = 2 > 1 \text{ konv.}$$

Konvergenca Alternirajočih vrst

$\pm \sum (-1)^n a_n, \quad a_n > 0$

- a_n pada
- $\lim_{n\to\infty} a_n = 0$

$\sum_{n=1}^{\infty} (-1)^n a_n$, $a_n > 0$, konv.

Leibnitzov kriterij: Če zaporedje a_n od nekje naprej pada proti 0, alternirajoča vrsta konvergira.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots$$

$a_n = \frac{1}{n}$; a_n pada? ✓

$\lim_{n\to\infty} a_n = 0$ ✓

$\} \Rightarrow$ vrsta konvergira

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n)!}{n^n} \qquad a_n = \frac{n!}{n^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)\cancel{n!}}{(n+1)^n \cancel{(n+1)}}}{\frac{\cancel{n!}}{n^n}} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n < 1 \quad \boxed{\text{pada}}$$

$\to 0$

$$\lim_{n\to\infty} \frac{n!}{n^n} = \lim \frac{n(n+1)(n-2)\ldots 2 \cdot 1}{n \cdot n \cdot n \ldots n \cdot n} = \underline{0}$$

$\shortparallel \quad \wedge \quad \wedge \quad \wedge \quad \wedge$

$1 \quad 1 \quad 1 \quad 1 \quad 1$

$\boxed{\text{konvergira}}$

limita števila

$0 \cdot \text{omejeno} = 0$

$0 \cdot \infty = \text{nedoločeno}$

DN $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4^n}{n!}$ R: konv.

FUNKCIJE

kodomena

$f: A \to B$

$x \mapsto f(x)$

domena ali D_f

$Z_f \subseteq B$

$Z_f = \{f(x) : x \in D_f\}$ → zaloga vrednosti, množica slik

D_f množica originalov

Določi D_f funkcije:

a) $f(x) = \underbrace{\ln(4-x^2) + \frac{1}{x^2-1}}$

kdaj je definirano?
(za katere x)

$\ln: \; 4-x^2 > 0$

$-: \; x^2-1 \neq 0$

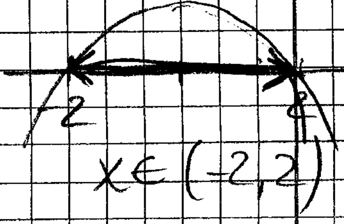
$D_f = \{x \in \mathbb{R}; 4-x^2 > 0 \text{ in } x^2-1 \neq 0\}$

$x^2-1 \neq 0$

$x \neq -1, 1$

$4-x^2 > 0$

$(2-x)(2+x) > 0$



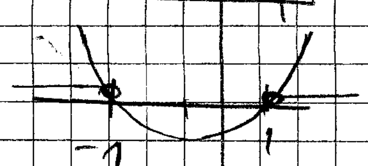
$x \in (-2, 2)$

$$\boxed{D_f: = (-2,2) \setminus \{-1, 1\}}$$

DN

$f(x) = x + \ln \frac{x-1}{x+1}$ $\quad R: (-\infty,-1) \cup (1,\infty)$

$f(x) = x + \sqrt{\frac{x-1}{x+1}}$ $\quad \frac{x-1}{x+1} \geq 0 \;/(x+1)^2$



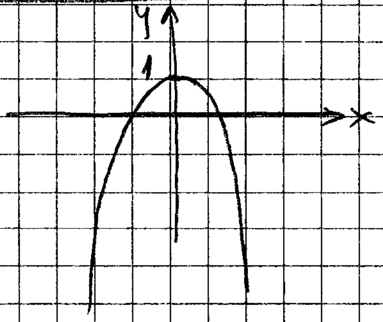
$\frac{x-1}{x+1} \geq 0 \Rightarrow (x-1)(x+1) \geq 0$

$x \neq -1$

$x \in (-\infty,-1] \cup [1,\infty)$

$Df: \underbrace{x \in (-\infty,-1) \cup [1,\infty)}$

Zaloga vrednosti in Df

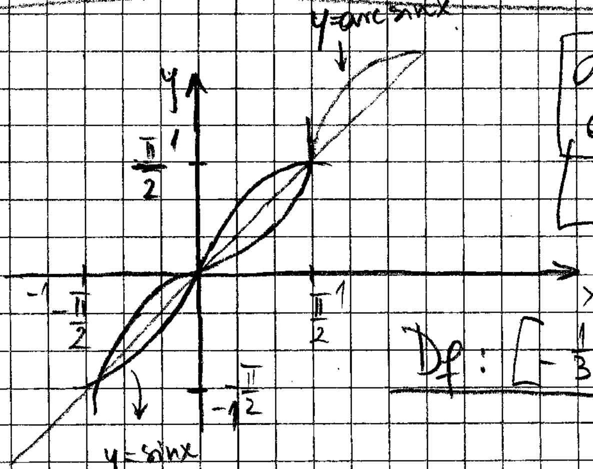


$f(x) = 1 - x^2$

$Df = (-\infty,\infty) = R$

$Zf = (-\infty,1]$

$f(x) = \arcsin \frac{2x}{x+1}$



arccosx
arctanx
DN

$D_{\arcsin x} = [-1,1]$

$Z_{\arcsin x} = [-\frac{\pi}{2}, \frac{\pi}{2}]$

$Df: [-\frac{1}{3}, 1]$

$-1 \leq \frac{2x}{x+1} \leq 1$

$x+1 \neq 0$

I. $-1 \leq \frac{2x}{x+1} \;/(x+1)^2$

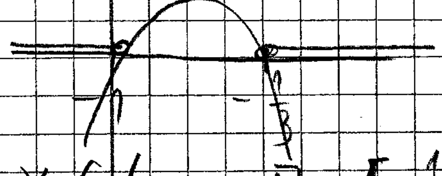
$-(x+1)^2 \leq 2x\,(x+1)$

$(x+1)(-(x+1) - 2x) \leq 0$

$(x+1)(-3x-1) \leq 0$

$x_1 = -1$

$x_2 = -\frac{1}{3}$



$x \in (-\infty,-1] \cup [-\frac{1}{3},\infty)$

PRESEK

$\{-1\} \cup [-\frac{1}{3}, 1]$

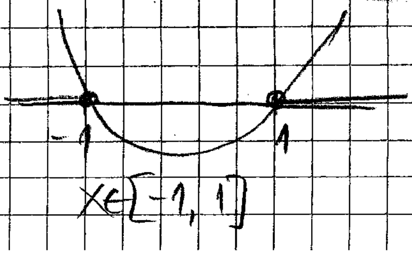
II. $\frac{2x}{x+1} \leq 1$

$2x(x+1) \leq (x+1)^2$

$(x+1)^2(2x - x - 1) \leq 0$

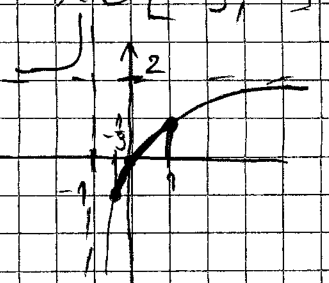
$(x+1)(x-1) \leq 0$

$x_1 = -1 \quad x_2 = 1$



$x \in [-1, 1]$

Z_f: $x \in \left[-\frac{1}{3}, 1\right] \Rightarrow \frac{2x}{x+1} \in [-1, 1] \Rightarrow \arcsin\frac{2x}{x+1}$



$\cancel{Z_f} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$Z_f = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

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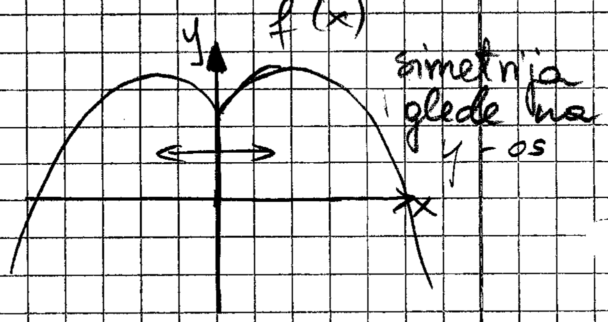
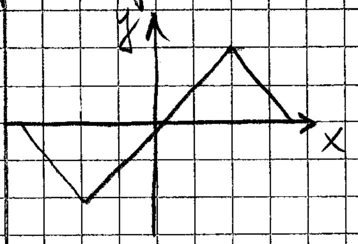
sodost / lihost funkcije:

25. 11. 201

soda funkcija: $f(-x) = f(x)$



liha funkcija: $f(-x) = -f(x)$



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ugotovi ali je funkcija liha ali soda:

a) $f(x) = \frac{\sin x}{x^3}$

$$f(-x) = \frac{\sin(-x)}{(-x)^3} = \frac{-\sin x}{-x^3} = \underline{\underline{\frac{\sin x}{x^3}}}$$

b) $f(x) = x + \sqrt{x^4 + x^6}$

$f(-x) = -x + \sqrt{x^4 + x^6}$ niti soda, niti liha

---

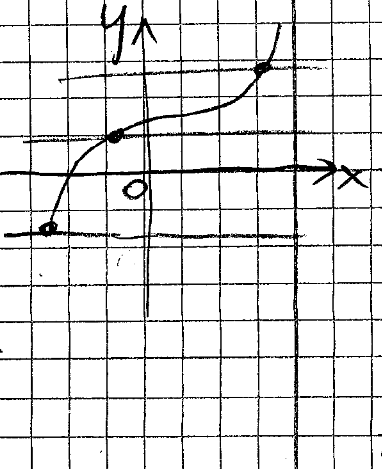
injektivnost / surjektivnost / bijektivnost

$*f: A \to B$

injektivna: dve različni števili imata različni sliki

$$(x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

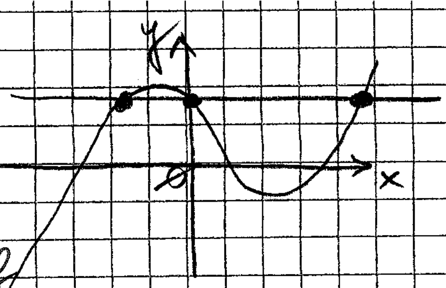
Vsaka vodoravnica $\swarrow$ v B seka graf **največ** 1x



* $f: A \rightarrow B$

surjektivna: $Z_f = B$

Vsaka vodoravnica v B seka graf vsaj 1x.



* bijektivna:

Vsaka vodoravnica v B seka graf natanko 1x.

Ugotovi, ali je funkcija $f(x) = \frac{2x}{1-x^2}$ inj., surj., bij., če:

a) $f: \mathbb{R} \rightarrow \mathbb{R}$

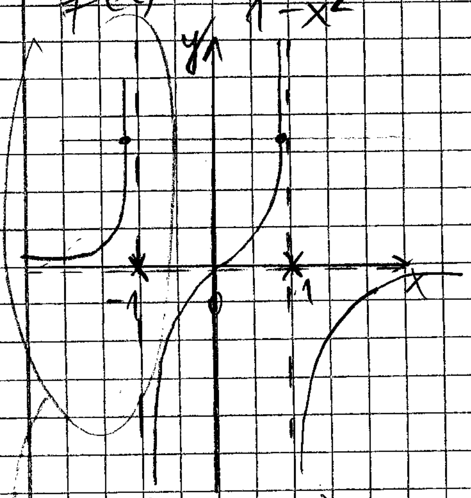
$f(x) = \frac{2x}{1-x^2}$

n: $x = 0$

p: $x^2 = 1 \Rightarrow x = \pm 1$

a: $\emptyset$ (st. števca < st. imenovalca)

$P_y(0,0)$

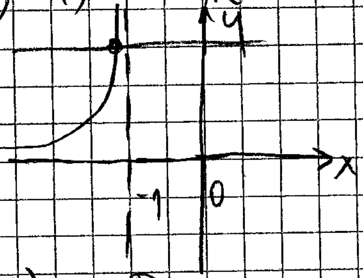


ni injektivna

je surjektivna

ni bijektivna

b) $f: (-\infty, -1) \rightarrow \mathbb{R}$



je injektivna

ni surjektivna

ni bijektivna

c) $f: (-1, 1) \rightarrow \mathbb{R}$

je injektivna

je surjektivna

je bijektivna

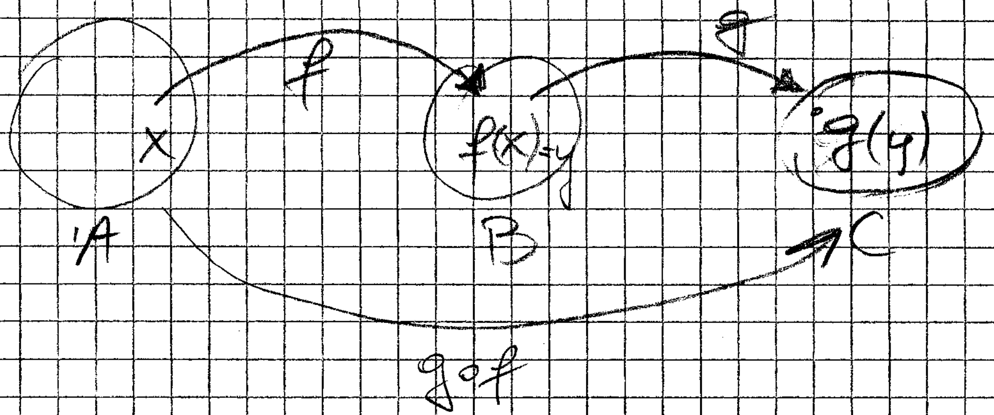
d) $f: (1, \infty) \rightarrow (-\infty, 0)$

je injektivna

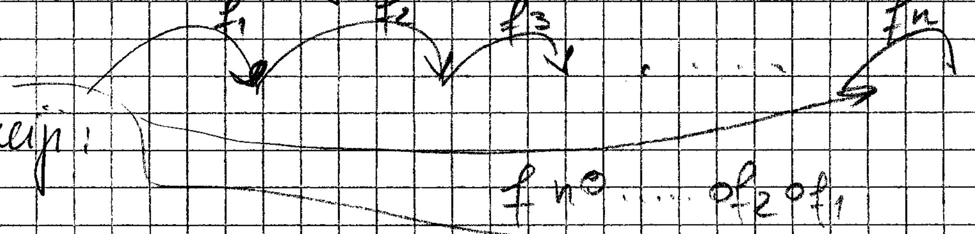
je surjektivna

je bijektivna

kompozitum / kompozicije funkcij



$z = g(y) = g(f(x)) = (g \circ f)(x)$ ← obratni vrstni red!



Dani sta funkciji:

$f(x) = x^2$

$g(x) = \sqrt{x}$

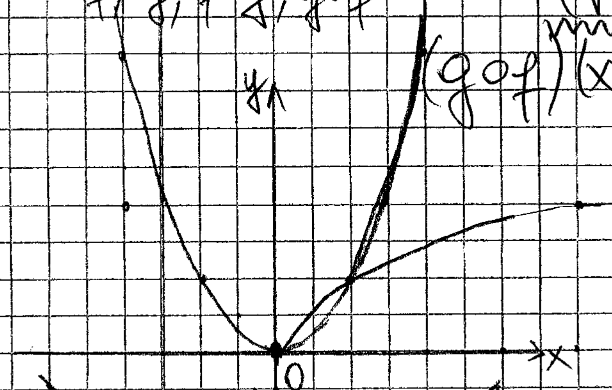
$(f \circ g)(x) = f(\boxed{g(x)}) = f(\sqrt{x}) =$

D_f, Z_f

$f, g, f \circ g, g \circ f$

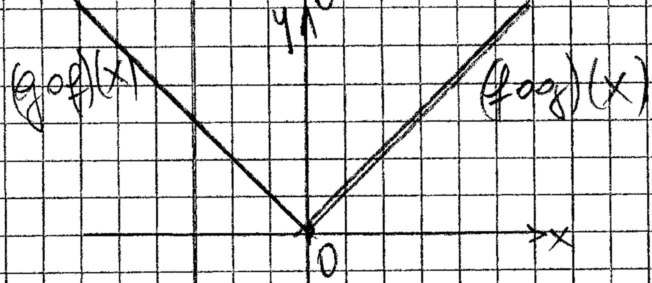
$= (\underbrace{\sqrt{x}}_{})^2 = x$

$(g \circ f)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = |x|$



$D_f = \mathbb{R}$ $\quad D_g = [0, \infty)$

$Z_f = [0, \infty)$ $\quad Z_f = [0, \infty)$



$D(f \circ g) = [0, \infty)$

$Z(f \circ g) = [0, \infty)$

$D(g \circ f) = \mathbb{R}$

$Z_f(g \circ f) = [0, \infty)$

Izračunaj kompozicije:

$f \circ g, g \circ f, f \circ f, g \circ g$

a) $f(x) = -1 + 2x$
$g(x) = 2 - 3x$

$(f \circ f)(x) = -1 + 2(-1 + 2x) = -3 + 4x$

$(f \circ g)(x) = -1 + 2(2 - 3x) = -1 + 4 - 6x = 3 - 6x$

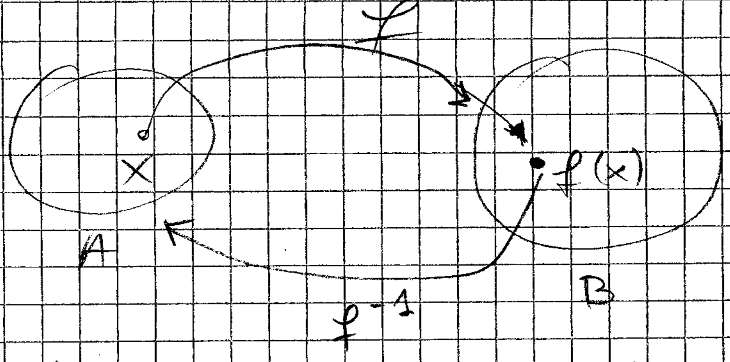
$(g \circ f)(x) = 2 - 3(-1 + 2x) = 2 + 3 - 6x = 5 - 6x$

$(g \circ g)(x) = 2 - 3(2 - 3x) = 2 - 6 + 9x = -4 + 9x$

DN $f(x)=\frac{1}{1+x^2}$

$g(x)=\sqrt{\frac{1-x}{x}}$

Inverzna funkcija



$(f\circ f^{-1})(x)=(f^{-1}\circ f)(x)$

Poišči inverzno funkcijo: $f(x)$:

a) $f(x)=1+atg(3x)$

$y=1+atg(3x)$

$x=1+atg(3y)$

$atg(3y)=x-1$

$3y=tg(x-1)$

$y=\frac{1}{3}tg(x-1)$

$f^{-1}(x)=\frac{1}{3}tg(x-1)$

b) $f(x)=e^x-e^{-x}$

$y=e^x-e^{-x}$

$x=e^y-e^{-y}$, $t=e^y>0$

$\boxed{x=e^y-\frac{1}{e^y}}$

$t_{1,2}=\frac{x\pm\sqrt{x^2+4}}{2}$

$x=t-\frac{1}{t}=\frac{t^2-1}{t}$

$-t^2+tx+1=0$

$t^2-tx-1=0$

$\boxed{t_1=\frac{x+\sqrt{x^2+4}}{2}}$

$\cancel{t_2=\frac{x-\sqrt{x^2+4}}{2}}<0$

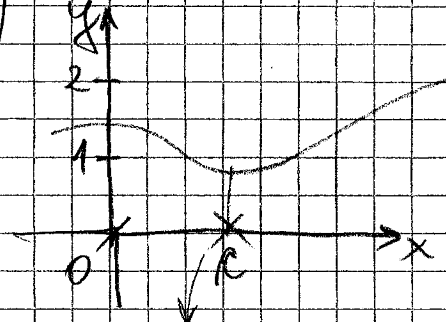
$e^y=\frac{x+\sqrt{x^2+4}}{2}$ /ln

$\boxed{y=\ln\frac{x+\sqrt{x^2+4}}{2}=f^{-1}(x)}$

Limite funkcij

f definirana na (a,b) razen morda v $c \in (a,b)$

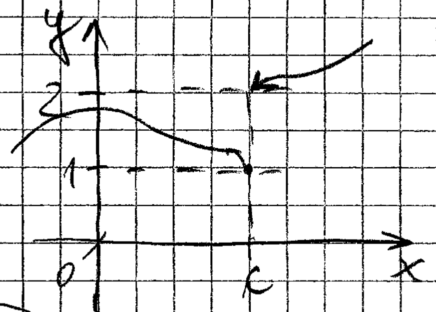
$$\boxed{\lim_{x\to c} f(x)}$$



točka ne vpliva na limito

$$\boxed{f(c) = 2}$$

$$\boxed{\lim_{x\to c} f(x) = 1}$$



$\lim f(x)$ ne obstaja;

leva limita:

$$\boxed{\lim_{x\uparrow c} f(x) = 1}$$

desna limita:

$$\boxed{\lim_{x\downarrow c} f(x) = 2}$$

Vemo:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^{-x} = \lim_{x\to -\infty}\left(1-\frac{1}{x}\right)^{x} = \lim_{x\to -\infty}\left(1-\frac{1}{x}\right)^{-x}$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1\ ;\quad \lim_{x\to\infty}\frac{\sin x}{x} = 0\ ;\quad \lim_{x\to\infty}\frac{p(x)}{e^{x}} = 0$$

Izračunaj limite:

a) $\lim_{x\to 2}\frac{3x^2-2x+1}{4x^2} = \frac{12-4+1}{16} = \frac{9}{16}$

b) $\lim_{x\to\infty}\frac{\sqrt{x^2+1}-\sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1}-\sqrt[5]{x^4+1}} \Big/ \frac{:x}{:x} = \lim_{x\to\infty}\frac{\sqrt{1+\frac{1}{x}}-\sqrt[3]{\cancel{\frac{1}{x}+\frac{1}{x^3}}}\to 0}{\sqrt[4]{1-\frac{1}{x}}-\sqrt[5]{\cancel{\frac{1}{x}+\frac{1}{x^2}}}\to 0} = \underline{\underline{1}}$

c) $\lim_{x\to 1}\frac{\sin 3x}{\cancel{x}} = \underline{\underline{\sin 3x}}$

d) $\lim_{x\to 0}\frac{\sin 3x}{x} \Big/ \frac{3}{3} = \frac{3\sin 3x}{3x} = \underline{\underline{3}}$

e) $\lim_{x\to 0}\frac{\sin ax}{\sin bx} \Big/ \frac{ax}{ax} = \lim_{x\to 0}\frac{ax\,\sin ax}{\sin bx \cdot ax} = \lim_{x\to 0}\frac{ax\ \cancel{bx}}{\sin bx\ \cancel{bx}} = \boxed{\frac{a\cancel{x}}{b\cancel{x}}}$

f) $\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x^2-1}\right)^{x^2} = \lim\limits_{x\to\infty}\left(\frac{x^2-1+2}{x^2-1}\right)^{x^2} = \lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{x^2-1}{2}}\right)^{x^2}$

$\lim\limits_{x\to\infty}\left(1+\frac{1}{\frac{\cancel{2m+1-1}}{\cancel{2}}}\right)^{2m+1} = \underline{\underline{e^2}}$

$\frac{x^2-1}{2}=2m$

$2m+1=x$

---

g) $\lim\limits_{x\to4}\frac{(\sqrt{1+2x}-3)}{(\sqrt{x}-2)} = \lim\limits_{x\to4}\frac{(\sqrt{1+2x}-3)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)} =$

$=\lim\limits_{x\to4}\frac{(\sqrt{1+2x}-3)(\sqrt{x}+2)}{(x-4)} = \lim\limits_{x\to4}\frac{(\sqrt{1+2x}-3)(\sqrt{x}+2)(\sqrt{1+2x}+3)}{(x-4)(\sqrt{1+2x}+3)} =$

$=\lim\limits_{x\to4}\frac{(1+2x-9)(\sqrt{x}+2)}{(x-4)(\sqrt{1+2x}+3)} = \lim\limits_{x\to4}\frac{2\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{(x-4)}(\sqrt{1+2x}+3)} = \frac{8}{6} = \boxed{\frac{4}{3}}$

$$\boxed{\text{h) DN} \quad \lim_{x\to1}\frac{x-\sqrt{x}}{\sqrt{x}-1} \quad R: 1}$$

i) $\lim\limits_{x\to1}\frac{x^m-1}{x^n-1}$

$x^n-1=(x-1)(x^{n-1}+x^{n-2}+\dots+x+1)$

$$\boxed{a+ag+\dots+ag^{n-1}=a\frac{1-g^n}{1-g}}$$

$\lim\limits_{x\to1}\frac{\cancel{(x-1)}(x^{m-1}+x^{m-2}+\dots+x^2+x+1)}{\cancel{(x-1)}(x^{n-1}+x^{n-2}+\dots+x^2+x+1)} = \frac{m}{n}$

$\sqrt[2]{\ }: (a-b)(a+b)$
$a^2 - b^2$

$\sqrt[3]{\ }: (a-b)(a^2+ab+b^2)$
$=a^3-b^3$

j) $\lim\limits_{x\to\infty}\left(\overset{a}{\sqrt[3]{(x+1)^2}}-\overset{b}{\sqrt[3]{(x-1)^2}}\right)\cdot$

$\cdot\frac{\left(\sqrt[3]{(x+1)^2}\right)^2+\sqrt[3]{(x+1)^2}\sqrt[3]{(x-1)^2}+\left(\sqrt[3]{(x-1)^2}\right)^2}{-//-}$

$=\lim\limits_{x\to\infty}\frac{(x+1)^2-(x-1)^2}{\underbrace{\sqrt[3]{(x+1)^4}+\sqrt[3]{(x+1)^2}\sqrt[3]{(x-1)^2}+\sqrt[3]{(x-1)^4}}} = \lim\limits_{x\to\infty}\frac{4x}{\ } \quad \frac{x}{x^{4/3}}$

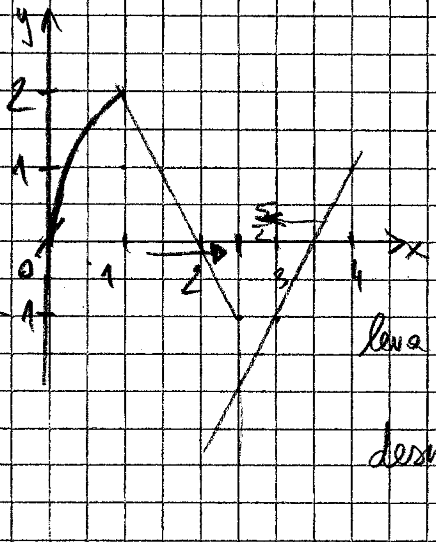
$\lim\limits_{x\to\infty} = 0$

4. V točkah, kjer funkcija ni zvezna, določi levo in desno limito:

$$f(x) = \begin{cases} 2\sqrt{x}; & 0 \leq x < 1 \\ 4-2x; & 1 \leq x < 5/2 \\ 2x-7; & \frac{5}{2} \leq x \leq 4 \end{cases}$$

možne nezveznosti

$x = 1$
$x = \frac{5}{2}$



$\boxed{x = \frac{5}{2}}$

leva lim: $\lim\limits_{x \uparrow \frac{5}{2}} f(x) = -1$

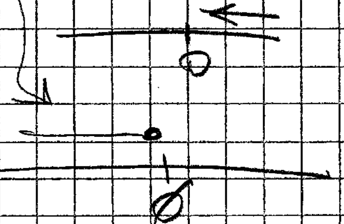
desna lim: $\lim\limits_{x \downarrow \frac{5}{2}} (f(x)) = -2$

Določi parameter a, tako, da bo funkcija f zvezna:

$$f(x) = \begin{cases} \frac{\sqrt[3]{x+1}-1}{x}; & x > 0 \\ a \cdot \operatorname{arctg}(1-x); & x \leq 0 \end{cases}$$

$f(x)$ bo zvezna v $x = 0$, če:

$$\lim_{x \uparrow 0} f(x) = \lim_{x \downarrow 0} f(x) = f(0)$$



L: $\lim\limits_{x \uparrow 0} f(x) = \lim\limits_{x \uparrow 0} = a \operatorname{arctg}(1-x) = = a \cdot \frac{\pi}{4}$

D: $\lim\limits_{x \downarrow 0} f(x) = \lim\limits_{x \downarrow 0} \frac{\sqrt[3]{x+1}-1}{x} \cdot \frac{\sqrt[3]{(x+1)^2}+\sqrt[3]{(x+1)}+1}{\text{——}\,''\,\text{——}} =$

$= \lim\limits_{x \downarrow 0} \frac{x+1-1}{x(\ldots)} = \lim\limits_{x \downarrow 0} \frac{1}{\sqrt[3]{(x+1)^2}+\sqrt[3]{(x+1)}+1} = \boxed{\frac{1}{3}}$

$a \cdot \frac{\pi}{4} = \frac{1}{3} = f(0)$

$\boxed{a = \frac{4}{3\pi}}$

$\sum_{n=1}^{\infty} \frac{2n}{(2n)!} \leftarrow a_n$ $\qquad \mathcal{D} = \lim_{n\to\infty} \frac{\frac{2(n+1)}{(2(n+1))!}}{\frac{2n}{(2n)!}} =$

$= \lim_{n\to\infty} \frac{(2n)!(2n+2)}{(2n+2)!\,2n} = \lim_{n\to\infty} \frac{\cancel{(2n)!}\,\cancel{(2n+2)}}{\cancel{(2n+2)}(2n+1)\cancel{(2n)!}\cdot 2n} = \lim_{n\to\infty} \frac{1}{2n(2n+1)}$

$= 0 < 1 \Rightarrow$ konvergira

|2004|

①

$\{(x,y)\in R^2;\ 2x-y+1<0 \text{ ali } x^2+y^2-2(x+y)\le 2\}$

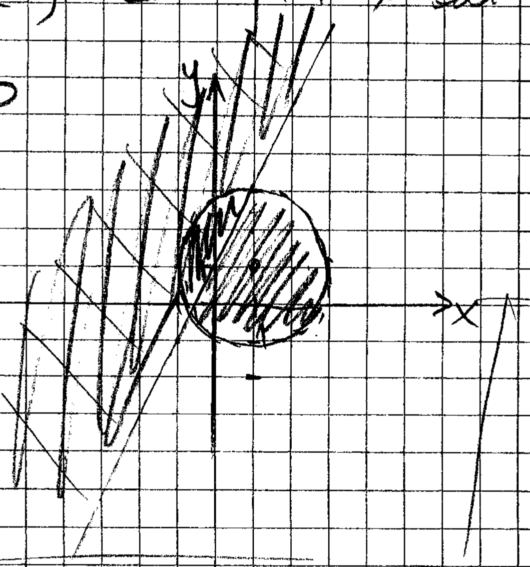
$2x-y+1<0$

$2x+1<y$

$x^2+y^2-2x-2y\le 2$

$(x-1)^2-1+(y-1)^2-1\le 2$

$(x-1)^2+(y-1)\le 4$



$\sum_{n=1}^{\infty} \frac{1}{\ln(n!)}$ $\qquad \frac{1}{\ln n}$

25.11.2010

$n < n!$

$\ln n < \ln(n!)$

$\frac{1}{\ln n} > \frac{1}{\ln(n!)}$

$|z-i| = |z+1|$ $\qquad z = x+iy$

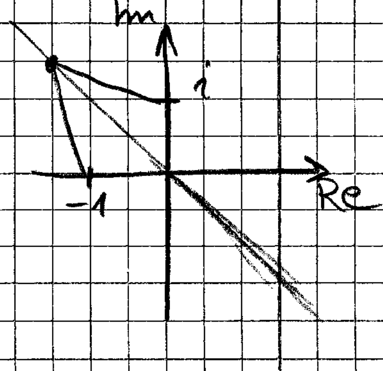
$|x+iy-i| = |x+iy+1|$

$\sqrt{x^2+(y-1)^2} = \sqrt{(x+1)^2+y^2}$ $\quad 2$

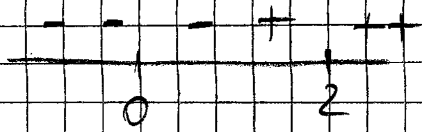
$\cancel{x^2}+\cancel{y^2}-2y+\cancel{1} = \cancel{x^2}+2x+\cancel{1}+\cancel{y^2}$

$y = -x$

$z = x - ix$



$|x-2| < x\cdot|x|$



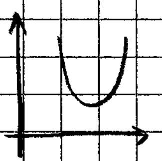
1.) $x<0$

$-x+2 < -x^2$

$x^2-x+2<0$

$D<0$

ni rešitve



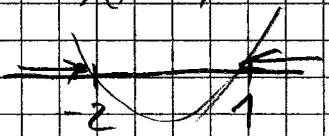
$\boxed{R:\ (1,\infty)}$

2.) $0\le x \le 2$

$-x+2 < x^2$

$x^2+x-2>0$

$(x-1)(x+2)>0$

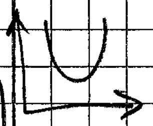


$(1,2)$

3.) $x\ge 2$

$x-2<x^2$

$x^2-x+2>0$



$[2,\infty)$

$|z|^2 + \frac{1}{\bar z} - \left(\mathrm{Re}(z)\right)^2 = 2$ $z=x+iy$

$\cancel{x^2}+y^2+\frac{(x+iy)}{(x-iy)(x+iy)} - \cancel{x^2} = 2$

$y^2+\frac{x+iy}{x^2+y^2}=2$

$y^2+\frac{x}{x^2+y^2}=2$ $\frac{y}{x^2+y^2}=0$

$y=0$

$\frac{1}{x}=2$

$\boxed{x=\frac{1}{2}}$

$a_n=\frac{1}{2}\left(a_{n+1}+\frac{3}{a_{n-1}}\right)$, $a_1=2$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{2}\left(a_{n-1}+\frac{3}{a_{n+1}}\right)\right)$ $a=\lim_{n\to\infty} a_n$

$a=\frac{1}{2}\left(a+\frac{3}{a}\right)\ /2a$ $a_n>0$

$2a^2=a^2+3$

$a^2=3$

$a_1=\sqrt{3}$

~~$a_2=-\sqrt{3}$~~

$$a_n - a_{n-1} =$$

$$= \frac{1}{2} a_{n-1} + \frac{3}{2a_{n-1}} - \frac{1}{2} a_{n-2} - \frac{3}{2a_{n-2}}$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \underbrace{\frac{4^n}{n!}}_{a_n} \qquad a_n = \frac{4^n}{n!}$$

padajoče:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{\cancel{4^n} \cdot 4}{(n+1)\cancel{n!}}}{\frac{\cancel{4^n}}{\cancel{n!}}} = \frac{4}{n+1} \leq 1 \quad \boxed{\text{za } n \geq 3}$$

konvergira

$$\lim_{n \to \infty} \frac{4^n}{n!} \qquad \frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot \ldots 4 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots n+1 \cdot n} \to 0$$

$$\sum_{n=1}^{\infty} \underbrace{\frac{(n!)^2}{(2n)!}}_{a_n} =$$

$$q = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \frac{\cancel{(2n)!}\cancel{(n!)^2}(n+1)^2}{\cancel{(n!)^2}(2n+2)(2n+1)\cancel{(2n)!}} =$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4} < 1 \quad \text{vrsta konvergira}$$

$$\lim_{n\to\infty} \frac{2^{n+1} + 3 \cdot 5^{2n+1}}{2 \cdot 3^{2n-1} (-1)^n 25^n} \Big/ \frac{5^{2n+1}}{5^{2n+1}} =$$

$$= \lim_{n\to\infty} \frac{\cancel{\frac{2^{n+1}}{5^{2n+1}}} + 3}{\cancel{2 \cdot \frac{3^{n-1}}{5^{2n+1}}} + (-1)^n \frac{1}{5}} \quad \begin{cases} -15 \\ 15 \end{cases}$$

2 stekališči, ni limite

$$\lim_{n\to\infty} \left(\frac{n^2+n}{n^2+n-1}\right)^{2n^2+2n} = \lim_{n\to\infty} \left(\frac{n^2+n-1}{n^2+n}\right)^{-2(n^2+n)} = (e^{-1})^{-2} = e^2$$

$$z^5 + \sqrt{\frac{3}{2}}\, z + i\sqrt{\frac{3}{2}}\, z = 0$$

$$z\left(z^4 + \sqrt{\frac{3}{2}} + i\sqrt{\frac{3}{2}}\right) = 0 \qquad \boxed{z^4 = -\frac{\sqrt{3}}{\sqrt{2}} - i\frac{\sqrt{3}}{\sqrt{2}}}$$

$$z_1 = 0$$

$$z_k = \sqrt[n]{r}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right)$$

$$r = \sqrt{\frac{3}{2} + \frac{3}{2}} = \sqrt{3}$$

$$\varphi = \operatorname{atg}\frac{-\sqrt{\frac{3}{2}}}{-\sqrt{\frac{3}{2}}} = \operatorname{atg} 1 = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

$$z_0 = \sqrt[8]{3}\left(\cos\frac{5\pi}{16} + i\sin\frac{5\pi}{16}\right)$$

$$\sum_{n=0}^{\infty}\frac{2^n a^n}{3^n} = \frac{3a}{2}$$

$a_1 = 1$

$q = \frac{2a}{3}$

$$\frac{a_1}{1-q} = \frac{3a}{2}$$

$$\frac{1}{1-\frac{2a}{3}} = \frac{3a}{2}$$

$$\frac{3}{(3-2a)} = \frac{3a}{2}$$

$$2 = 3a - 2a^2$$

$$2a^2 - 3a + 2 = 0$$

$$a_{1,2} = \frac{3 \pm \sqrt{9-16}}{4} \quad D < 0$$ nima realnih ničel

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2005/4

$$f(x) = \sqrt{2-x}$$

$$g(x) = \frac{x+1}{x-1}$$

$$(f \circ g)(x) = f(g(x)) = \sqrt{2 - \frac{x+1}{x-1}} = \sqrt{\frac{x-3}{x-1}}$$

$$(g \circ f)(x) = g(f(x)) = \frac{(\sqrt{2-x}+1)}{(\sqrt{2-x}-1)}\frac{(\sqrt{2-x}+1)}{(\sqrt{2-x}+1)} = \frac{(\sqrt{2-x}+1)^2}{1-x}$$

$2 - x \geq 0$

$x \leq 2$

$x \neq 1$

$$\boxed{D(g \circ f) = (-\infty, 1) \cup (1, 2] \Rightarrow (-\infty, 2] - \{1\}}$$

$$\frac{x-3}{x-1} \geq 0 \quad / (x-1)^2$$

$$(x-3)(x-1) \geq 0$$

$x_1 = 1$

$x_2 = 3$

$$\boxed{D(f \circ g) = (-\infty, 1) \cup [3, \infty)}$$

$$\big||x-1|\big| \geq |x| + x$$

$x=1$
$x=0$

$- - \quad - + \quad + +$
$0 \quad 1$

$$\sum_{n=1}^{\infty} \frac{2^{2n+1}}{n\,3^n}$$

korenski kriterij

$4^n \cdot 2$

$$g = \lim_{n\to\infty} \sqrt[n]{\frac{2^{2n+1}}{n\,3^n}} = \lim_{n\to\infty} \frac{4\,\cancel{\sqrt[n]{2}}}{\sqrt[n]{n}\cdot 3} = \frac{4}{3} > 1 \quad \underline{\underline{\text{divergira}}}$$

$\to 1$

$\to 1$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

$$\lim_{n\to\infty} \sqrt[n]{c} = 1$$

DN:

$f(x) = \text{arctg}\,\frac{1+x}{1-x}$ R: $\frac{1}{1+x^2}$

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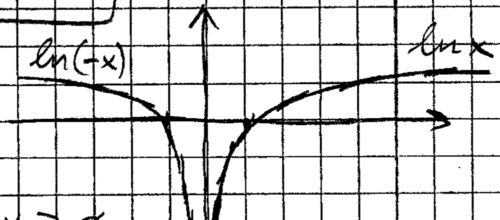
h) $f(x) = \ln|x| = \begin{cases} \ln x\,; & x \geq 0 \\ \ln(-x)\,; & x < 0 \end{cases}$



$$f'(x) = \begin{cases} \frac{1}{x}\,; & x \geq 0 \\ \frac{1}{(-x)}(-1)\,, & x < 0 \end{cases} = \begin{cases} \frac{1}{x}\,; & x \geq 0 \\ \frac{1}{x}\,; & x < 0 \end{cases} = \boxed{\frac{1}{x}}$$

i) $f(x) = \ln\left(\ln^2(\ln^3 x)\right)$

$\ln\left(\ln^2(\ln x)^3\right)$

$\ln\left(\ln(\ln x)^3\right)^2$

$\ln\left(3\ln(\ln x)\right)^2$

$2\ln\left(3\ln(\ln x)\right)$

$$f'(x) = 2\,\frac{1}{3\ln(\ln x)}\left(3\ln(\ln x)\right)'$$

$$f'(x) = \frac{2}{3\ln(\ln x)}\left(\frac{3}{\ln x}\right)(\ln x)'$$

$$f'(x) = \frac{2\,\cancel{3}}{\cancel{3}\ln(\ln x)\ln x}\,\frac{1}{x}$$

---

j) $f(x) = \ln\sqrt{\frac{e^{2x}}{1+e^{2x}}} = \frac{1}{2}\ln\frac{e^{2x}}{1+e^{2x}} = \frac{1}{2}\left(\ln e^{2x} - \ln(1+e^{2x})\right)$

$$= \frac{1}{2}\left(2x - \ln(1+e^{2x})\right) = x - \frac{\ln(1+e^{2x})}{2}$$

$$f'(x) = 1 - \frac{1}{\cancel{2}(1+e^{2x})}\,\cancel{2}e^{2x} = 1 - \frac{e^{2x}}{1+e^{2x}} = \boxed{\frac{1}{1+e^{2x}}}$$

k) $f(x)=x^{1/x}$

$$(x^r)'=rx^{r-1}$$

$$(a^x)'=a^x\ln a$$

$f(x)=e^{\ln x^{\frac{1}{x}}}=e^{1/x\ln x}$

$\boxed{x=e^{\ln x}}$; e in ln sta inverzni funkciji

$f'(x)=e^{1/x\ln x}\left(\underset{\overset{\|}{-\frac{1}{x^2}}}{-x^{-2}}\ln x+\frac{1}{x}\cdot\frac{1}{x}\right)$

$$f'(x)=\frac{1}{x^2}x^{1/x}(1-\ln x)=x^{\frac{1}{x}-2}(1-\ln x)$$

Odvajaj implicitno podano funkcijo

$\ln(x^2+y^2)=\operatorname{arctg}\frac{y}{x}$

$\frac{1}{x^2+y^2}(x^2+y^2)'=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{y}{x}\right)'$

$\frac{1}{x^2+y^2}(2x+2yy')=\frac{\cancel{x^2}}{y^2+x^2}\cdot\frac{y'x-y}{\cancel{x^2}}\ /(x^2+y^2)$

$2x+2yy'=y'x-y$

$2yy'-y'x=-2x-y$

$y'(2y-x)=-2x-y$

$y'=\frac{-(2x+y)}{2y-x}\ \boxed{=\frac{2x+y}{x-2y}}$

EKSPLICITNA

$y=kx+n$; $y=2x+1$

IMPLICITNA

$2x+1-y=0$

$y^2x-\sin x=\cos y$

y zadosce enacbi

y funkcija

L'Hospitalovo pravilo :

Če računamo limito ulomka nedoločene oblike $\frac{0}{0},\frac{\infty}{\infty},-\frac{\infty}{\infty}$ velja:

$$\lim_{x\to\cancel{0}}\frac{f(x)}{g(x)}=\lim_{x\to\cancel{0}}\frac{f'(x)}{g'(x)}$$

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1$$

D.N.

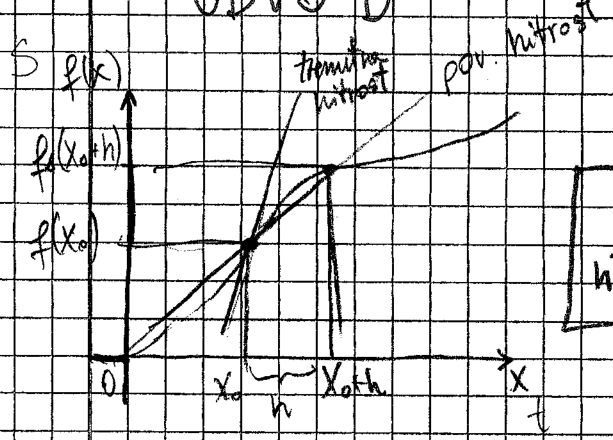
$\lim_{x\to 0}\frac{x^m-1}{x^n-1}$, $\boxed{R=\frac{m}{n}}$

a) $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{0}{0}\Rightarrow\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$

$\lim_{x\to 0}\frac{2\arcsin x}{3x}=\frac{0}{0}=\frac{\frac{2}{\sqrt{1-x^2}}}{3}=\frac{2}{3}$

6.12.2011

ODVOD



$$k = \frac{\Delta y}{\Delta x} = \frac{f(x_0+h) - f(x_0)}{h}$$

$$\boxed{\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0)}$$

odvod v tocki x_0

$c' = 0$

$x' = 1$

$(x^r)' = rx^{r-1}\ ;\ r \in R$

$(\ln x)' = \frac{1}{x}$

$(\log_a x)' = \frac{1}{x \ln a}$

$(e^x)' = e^x$

$(a^x)' = a^x \cdot \ln a$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(tg x)' = \frac{1}{\cos^2 x} = 1 + tg^2 x$

$(ctg x)' = -\frac{1}{\sin^2 x} = -1 - ctg^2 x$

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

$(arctg x)' = \frac{1}{1+x^2}$

$(arcctg x)' = -\frac{1}{1+x^2}$

Pravila:

$(c \cdot f(x))' = 0 + cf'(x) = c \cdot f'(x)$

$(c + f(x))' = f'(x)$

$(f(x) + g(x))' = f'(x) + g'(x)$

$(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x)$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

$(f(g(x)))' = f'\cancel{(x)}(g(x)) \cdot g'(x)$

$\Downarrow$

$(f \circ g)(x)$

① odvajaj naslednje funkcije

a) $f(x) = x\sqrt{1-x^2} = x(1-x^2)^{1/2}$

$f'(x) = (1-x^2)^{1/2} + x\left(+\frac{1}{2}\right)(1-x^2)^{-1/2} \cdot \underline{(1-x^2)'}$

$(1-x^2)^{1/2} + \frac{1}{\cancel{2}}x(1-x^2)^{1/2}(-\cancel{2}x)$

$$f'(x) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = \frac{1-x^2-x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}$$

b) $f(x) = \frac{x^2-5x-1}{x^3}$

$$f'(x) = \frac{(2x-5)x^3-(x^2-5x-1)3x^2}{(x^3)^2} = \frac{x^2\left((2x-5)\cdot x - 3(x^2-5x-1)\right)}{x^6} =$$

$$= \frac{(2x-5)x-3(x^2-5x-1)}{x^4} = \frac{2x^2-5x-3x^2+15x+3}{x^4} =$$

$$= \frac{-x^2+10x+3}{x^4}$$

c) $f(x) = \arccos\frac{1}{x}$

$$f'(x) = -\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot \left(\frac{-1}{x^2}\right) = \frac{1}{x^2\sqrt{1-\left(\frac{1}{x}\right)^2}} = \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}} = \frac{|x|}{x^2\sqrt{x^2-1}} =$$

$$f'(x) = \frac{\cancel{|x|}}{\cancel{|x|}\,|x|\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$$

d) $f(x) = \ln^3(x^2) = (\ln x^2)^3 = (2\ln x)^3 = 8(\ln x)^3$

$$f'(x) = 8\;3(\ln x)^2 \cdot \frac{1}{x} = 24\,\frac{(\ln x)^2}{x}$$

e) $f(x) = \sqrt{\frac{x^2-1}{x^2+1}}$

$$f'(x) = \frac{2x(x^2+1)-(x^2-1)2x}{\left(2\sqrt{\frac{x^2-1}{x^2+1}}\right)(x^2+1)^2} = \frac{2x(\cancel{x^2}+1-\cancel{x^2}+1)}{2\left(\sqrt{\frac{x^2-1}{x^2+1}}\right)(x^2+1)^2} = \frac{2\cancel{4}x}{\cancel{2}\sqrt{\frac{x^2-1}{x^2+1}}(x^2+1)^2}$$

$$f'(x) = \frac{2x}{\sqrt{\frac{x^2-1}{x^2+1}}\,(x^2+1)^2} = \frac{2x\sqrt{x^2+1}}{\sqrt{x^2-1}\,(x^2+1)^2} = \frac{2x}{\sqrt{(x^2+1)^3}\left(\sqrt{x^2-1}\right)}$$

f) $f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$

$f'(x) = \frac{(\cos x - \sin x)(\sin x + \cos x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$

$= \frac{-(\cos x - \sin x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2} =$

~~sin 2x = 2sin~~

$\sin(2x) = 2\sin x\cos x$

$\cos(2x) = \cos^2 x - \sin$

$= \frac{-\cos^2 x + 2\cancel{\sin x \cos x} - \sin^2 x - \sin^2 x - \cancel{2\sin x \cos x} - \cos^2 x}{(\sin x - \cos x)^2}$

$\frac{-2(\cos^2 x + \sin^2 x)\Rightarrow 1}{(\sin x - \cos x)^2} = \frac{-2}{\sin^2 x - 2\sin x\cos x + \cos^2 x} = \frac{-2}{1 - \sin 2x} = \frac{2}{\sin 2x -}$

g) $f(x) = \operatorname{arctg}\frac{1+x}{1-x}$ DN

R: $f'(x) = \frac{1}{1+x^2}$

h) $f(x) = \ln|x| = \begin{cases} \ln x; & x>0 \\ \ln(-x); & x<0 \end{cases} = \begin{cases} \frac{1}{x}; & x>0 \\ \frac{1}{-x}(-1), & x<0 \end{cases} = \frac{1}{x}$

i)

$f(x) = \ln(\ln^2(\ln^3 x))$

$= \ln(\ln^2(\ln x)^3)$

$\ln(3\ln(\ln x))^2$

$2\ln(3\ln(\ln x))$

$f'(x) = 2\frac{1}{3\ln(\ln x)}\cdot\frac{\cancel{3}}{\ln x}\cdot\frac{1}{x}$

$f'(x) = \frac{2}{\ln(\ln x)\cdot\ln x}\cdot\frac{1}{x}$

j) $f(x) = \ln\sqrt{\frac{e^{2x}}{1+e^{2x}}} = \ln\left(\frac{e^{2x}}{1+e^{2x}}\right)^{1/2} = \frac{1}{2}\left(\ln(e^{2x}) - \ln(1+e^{2x})\right)$

$\frac{1}{2}(2x - \ln(1+e^{2x})) = x - \frac{1}{2}\ln(1+e^{2x})$

$f'(x) = 1 - \cancel{\frac{1}{2}}\frac{1}{(1+e^{2x})}\cancel{2}e^{2x} = 1 - \frac{e^{2x}}{1+e^{2x}} = \frac{1+\cancel{e^{2x}}-\cancel{e^{2x}}}{1+e^{2x}} = \frac{1}{1+e^{2x}}$

k) $f(x) = x^{1/x}$

$= e^{\ln x^{1/x}} = e^{\frac{1}{x}\ln x}$

$(f^{-1}\circ f)(x) = (f\circ f^{-1})(x) = x$ $\ln e^x = x$

$e^{\ln x} = x$

$f'(x) = e^{\frac{1}{x}\ln x}\left(-\frac{1}{x^2}\ln x + \frac{1}{x}\cdot\frac{1}{x}\right) = e^{\frac{1}{x}\ln x}\left(-\frac{1}{x^2}\ln x + \frac{1}{x^2}\right) =$

$= e^{\frac{1}{x}\ln x}\frac{1}{x^2}(1-\ln x) = x^{1/x}\frac{1}{x^2}(1-\ln x) = x^{\frac{1}{x}-2}(1-\ln x)$

② odvajaj implicitno podano funkcijo:

EKSPLICITNA

$y = f(x) = \ldots$

npr.: $f(x) = x\sin x - 2$

IMPLICITNA

npr.: $y^2 \cdot \sin x = 2xy - 3$

$f(x)$ oz. y nastopa v enačb

$$\ln(x^2+y^2) = \operatorname{atg}\frac{y}{x}$$

$$\frac{1}{x^2+y^2}\cdot(2x+2yy') = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{y'x-y\cdot 1}{x^2}$$

$$\frac{2x+2yy'}{x^2+y^2} = \frac{\cancel{x^2}}{x^2+y^2}\cdot\frac{y'x-y}{\cancel{x^2}}$$

$$2x+2yy = y'x-y$$

$$2yy'-y'x = -2x-y$$

$$y'(2y-x) = -2x-y$$

$$y' = \frac{-(2x+y)}{2y-x} = \underline{\underline{\frac{2x+y}{x-2y}}}$$

L'Hospitalovo pravilo

Če računamo limito ulomkov nedoločene oblike $\frac{0}{0}$, $\frac{\infty}{\infty}$ ali $\frac{-\infty}{\infty}$, velja:

$$\lim_{x\to a}\frac{f(x)}{g(x)} \underset{\uparrow}{=} \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

$\frac{0}{0}, \frac{\infty}{\infty}, -\frac{\infty}{\infty}$

③ Z uporabo L'Hospitalovega pravila izračunaj limite

a) $$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sin x}{2x} = \lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}$$

$\boxed{\frac{0}{0}}$ $\boxed{\frac{0}{0}}$

b) $\lim_{x\to 1}\frac{x^m-1}{x^n-1}$ R: $\frac{m}{n}$ DN

c) $$\lim_{x\to 0}\frac{2\arcsin x}{3x} = \lim_{x\to 0}\frac{\frac{2}{\sqrt{1-x^2}}}{3} = \frac{2}{3}$$

e) $$\lim_{x\to\infty}\frac{x^2}{e^{2x}} = 0$$

$$\lim_{x\to\infty}\frac{x^2}{e^{-2x}} \underset{\uparrow}{=} \lim_{x\to\infty}x^2e^{2x} = \infty$$

$\boxed{\frac{\infty}{0}}$

④ Pokaži, da je razlika funkcij odsekoma konstantna in jo določi

$f(x) = \text{arctg}\,\frac{1}{x}$

$g(x) = \text{arctg}\,\frac{x+1}{x-1}$

- (odsekoma) konstantno

$f(x) - g(x) = \text{konst} \qquad (\ )' = 0$

$$\left(\text{arctg}\,\frac{1}{x} - \text{arctg}\,\frac{x+1}{x-1}\right)' = -\frac{1}{\left(1+\left(\frac{1}{x}\right)^2\right)x^2} - \frac{1}{1+\left(\frac{x+1}{\cancel{x-1}}\right)^2}\,\frac{\cancel{x-1}-\cancel{x}-1}{(\cancel{x-1})^2} =$$

$$= \frac{2}{1+\left(\frac{x+1}{x-1}\right)^2 (x-1)^2} - \frac{1}{x^2+1} = \frac{2}{(x-1)^2+(x+1)^2} - \frac{1}{x^2+1} =$$

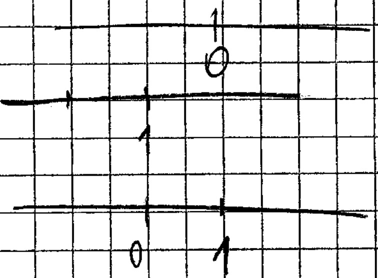
$$= \frac{2}{x^2-\cancel{2x}+1+x^2+\cancel{2x}+1} - \frac{1}{x^2+1} = \frac{\cancel{2}\ 1}{\cancel{2}(x^2+1)} - \frac{1}{x^2+1} = 0$$

- $f(x) - g(x) = ?$

$f(x)$ ni definirana v

$g(x)$ -||-

$f(x) - g(x)$ -||-



$(-\infty, 0),\ (0,1),\ (1,\infty)$ na vsakem določimo posebej

I. $(-\infty, 0)$: $x = -1$ ← izberemo

$$\left.\begin{aligned} f(-1) &= \text{atg}(-1) = -\frac{\pi}{4} \\ g(-1) &= \text{atg}(0) = 0 \end{aligned}\right\} \quad f(x) - g(x) = -\frac{\pi}{4} - 0 = -\frac{\pi}{4}$$

II. $(0,1)$: $\lim\limits_{x\downarrow 0} f(x) = \lim\limits_{x\downarrow 0} \text{arctg}\left(\frac{1}{x}\right)^{\nearrow\infty} = \frac{\pi}{2}$

$\lim\limits_{x\downarrow 0} g(x) = \lim\limits_{x\downarrow 0} \text{arctg}\,\frac{1+x}{1-x} = \text{atg}(-1) = -\frac{\pi}{4}$

$f(x) - g(x) = \frac{\pi}{2} - \left(-\frac{\pi}{4}\right) = \frac{3\pi}{4}$

III. $(1,\infty)$

$$\left.\begin{aligned} \lim_{x\downarrow 1} f(x) &= \lim_{x\downarrow 1} \text{atg}\,\frac{1}{x} = \frac{\pi}{4} \\ \lim_{x\downarrow 1} g(x) &= \lim_{x\downarrow 1} \text{atg}\left(\frac{x+1}{x-1}\right)_{\searrow\infty} = \frac{\pi}{2} \end{aligned}\right\} f(x) - g(x) = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$$

$$R: f(x) - g(x) = \begin{cases} -\frac{\pi}{4}, & x \in (-\infty, 0) \cup (1, \infty) \\ \frac{3\pi}{4}, & x \in (0, 1) \end{cases}$$

---

⑤ poišči drugi odvod funkcije

a) $f(x) = e^{\sin x} \cdot \cos x$

$$f'(x) = (e^{\sin x})' \cos x + e^{\sin x} \cdot (\cos x)' = \cos x\, e^{\sin x} \cdot \cos x - e^{\sin x}(-\sin x) =$$

$$= \cos^2 x\, e^{\sin x} - e^{\sin x}(\sin x) = e^{\sin x}(\cos^2 x - \sin x)$$

$$f''(x) = \cos x\, e^{\sin x}(\cos^2 x - \sin x) + e^{\sin x}(-2\sin x(-\cos x) - \cos x)$$

$$= \cos x\, e^{\sin x}(\cos^2 x - \sin x - 2\sin x - 1) = \underline{\cos x\, e^{\sin x}(\cos^2 x - 3\sin x - 1)}$$

b) $f(x) = x\sqrt{1+x^2}$

$$\boxed{R: f''(x) = \frac{x(2x^2+3)}{(1+x^2)^{3/2}}}$$ DN

c) $y x = \ln(1+y)$

$$y'x + y = \frac{1}{1+y} \cdot y'$$

$$\frac{1}{1+y} y' - y'x = y$$

$$y'\left(\frac{1}{1+y} - x\right) = y$$

$$y' = \frac{y}{\frac{1}{1+y} - x} =$$

$$y' = \frac{y(1+y)}{1 - x(1+y)}$$

$$y'' = \frac{(y' + 2yy')(1 - x - xy) - (y + y^2)(-1 - y - xy')}{(1 - x(1+y))^2}$$

⑥ poišči n-ti odvod funkcije $f(x) = e^{-3x}$

$$f(x) = e^{-3x}$$
$$f'(x) = -3e^{-3x}$$
$$f''(x) = (-3)^2 e^{-3x}$$
$$f'''(x) = (-3)^3 e^{-3x}$$

$$f^{(n)}(x) = e^{-3x}(-3)^n$$

⑦ Poišči n-ti odvod funkcije

$f(x) = \frac{1}{x^2 - a^2}$ $(a \in \mathbb{R})$, tako da jo zapišeš s parcialnimi ulomki.

$$f(x) = \frac{A}{x-a} + \frac{B}{x+a} = A(x-a)^{-1} + B(x+a)^{-1}$$

$$f'(x) = -A(x-a)^{-2} - B(x+a)^{-2}$$

$$R: f^{(n)}(x) = \frac{1}{2a}(-1)^n \cdot n \left((x-a)^{-n-1} \cancel{8}8a\right)^{-1}$$

13.12.2010

① Poišči n-ti odvod funkcije $\boxed{f(x) = \frac{1}{x^2-a^2}}$, tako da jo zapišeš s parcialnimi ulomki.

$a \in R$

$$f(x) = \frac{1}{x^2-a^2} = \frac{A}{x-a} + \frac{B}{x+a} = \frac{A(x+a)+B(x-a)}{(x-a)(x+a)} =$$

$$1 = A(\underline{x}+a) + B(\underline{x}-a)$$

x^0: $1 = Aa - Ba$ $\Rightarrow$ $\frac{1}{a} = A - B$

x^1: $0 = A + B$ $\quad$ $0 = A + B$

$$\frac{1}{a} = 2A$$

$$\boxed{A = \frac{1}{2a}} \qquad \boxed{B = -\frac{1}{2a}}$$

$$= \frac{\frac{1}{2a}}{x-a} - \frac{\frac{1}{2a}}{x+a}$$

$$f(x) = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right)$$

$$f'(x) = 0 + \frac{1}{2a}\left(-(x-a)^{-2}\cdot 1 - (-(x+a)^{-2})\cdot 1\right)$$

$$f''(x) = 0 + \frac{1}{2a}\left(-(x-a)^{-3}(-2) - (-(-2)(x+a)^{-3}\right)$$

$$f'''(x) = 0 + \frac{1}{2a}\left(-(-2)(-3)(x-a)^{-4} - (-(-2))(-3)(x+a)^{-4}\right)$$

$$f^{(n)}(x) = \frac{1}{2a}\left((-1)^n n!\,(x-a)^{-(n+1)} - (-1)^n n!\,(x-a)^{-(n+1)}\right) =$$

$$f^{(n)}(x) = \frac{1}{2a}(-1)^n \cdot n!\left((x-a)^{-(n+1)} - (x+a)^{-(n+1)}\right)$$

---

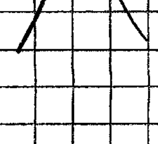
② Določi a in b, da bo funkcija zvezno odvedljiva:

$$f(x) = \begin{cases} e^x + x^2 + 1, & x < \underline{0} \\ ax + b; & x \geq \underline{\underline{0}} \end{cases}$$

} možni točki nezveznosti

zveznost:
- nepretrganost v vsaki točki

zvezna odvedljivost:
- odvod je zvezna funkcija



f(x) = zvezna,
NI odvedljiva

f(x) mora biti zvezna:

$\boxed{x=0}$

$$\lim_{x\uparrow 0} f(x) = \lim_{x\downarrow 0} f(x) = f(0)$$

$$\lim_{x\uparrow 0} (e^x + x^2 + 1) = \lim_{x\downarrow 0} (ax+b) = b$$

$$2 = b = b$$

$$\boxed{b=2}$$

$$f'(x) = \begin{cases} e^x + 2x, & x < 0 \\ a, & x \geq 0 \end{cases} \Bigg\} \quad f'(x) \text{ je zvezna funkcija}$$

$\boxed{x=0}$

$$\lim_{x\uparrow 0} f'(x) = \lim_{x\downarrow 0} f'(x) = f'(0)$$

$$\lim_{x\uparrow 0} (e^x + 2x) = \lim_{x\downarrow 0} (a) = a$$

$$1 = a = a$$

$$\boxed{a=1}$$

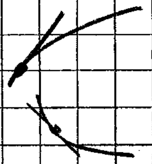
R: $a=1$
$b=2$

$$f(x) = \begin{cases} e^x + x^2 + 1, & x < 0 \\ x + 2; & x \geq 0 \end{cases}$$

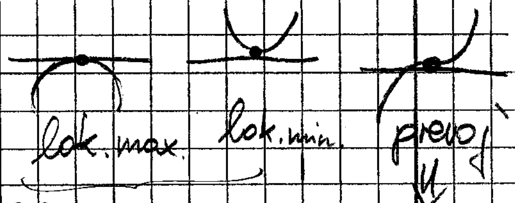
③ Določi intervale, kjer je funkcija $f(x) = \frac{2x}{1+x^2}$ naraščujoča, in intervale, kjer je f(x) padajoča

$f'(x) > 0$: f(x) narašča

$f'(x) < 0$: f(x) pada



$f'(x) = 0$ x je stacionarna točka



lokalna ekstrema

$$f(x) = \frac{2x}{1+x^2}$$

$$f'(x) = \frac{2(1+x^2) - 2x(2x)}{(1+x^2)^2} = \frac{2+2x^2-4x^2}{(1+x^2)^2} =$$

$$f'(x) = \frac{-2x^2+2}{(1+x^2)^2}$$

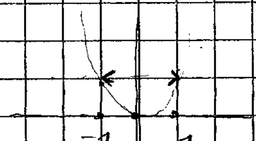
$$f'(x) = -2x^2 + 2 > 0$$

$$2x^2 < 2$$

$$x^2 < 1$$

$$x_1 < -1$$

$$x_2 < 1$$



$x \in (-1, 1) \Rightarrow$ narašča

$x \in (-\infty, -1) \cup (1, \infty) \Rightarrow$ pada

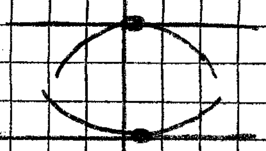
lokalni ekstremi:

$f'(a)=0$; a je stacionarna točka

$f''(a)<0$; v točki a je lokalni max.

$f''(a)>0$; v točki a je lokalni min.

$f''(0)=0$; lahko bi bil prevoj (tega ne bomo delali)



⑭ Poišči in klasificiraj vse lokalne ekstreme funkcije:

a) $f(x)=x\ln x$

iščemo stacionarne točke:

lokalni ekstremi so tisti jih najdemo z odvodi.

$f'(x)=\ln x+x\frac{1}{x}=\ln x+1$

$f'(x)=0$

$\ln x+1=0$

$\ln x=-1$

$\underline{\underline{x=e^{-1}}} \Rightarrow$ edina stacionarna točka

$f''(x)=\frac{1}{x}$

$f''(e^{-1})=\frac{1}{e^{-1}}=e>0 \Rightarrow$ v točki $x=e^{-1}$ je lokalni minimum

b) $f(x)=x^2e^{-x}$

$f'(x)=2xe^{-x}+x^2e^{-x}(-1)$

$e^{-x}(2x-x^2)=0$

$2x-x^2=0$

$x(2-x)=0$

$x_1=0$

$x_2=2$

$f''(x)=-e^{-x}(2x-x^2)+(2-2x)e^{-x}$

$e^{-x}(-(2x-x^2)+2-2x)$

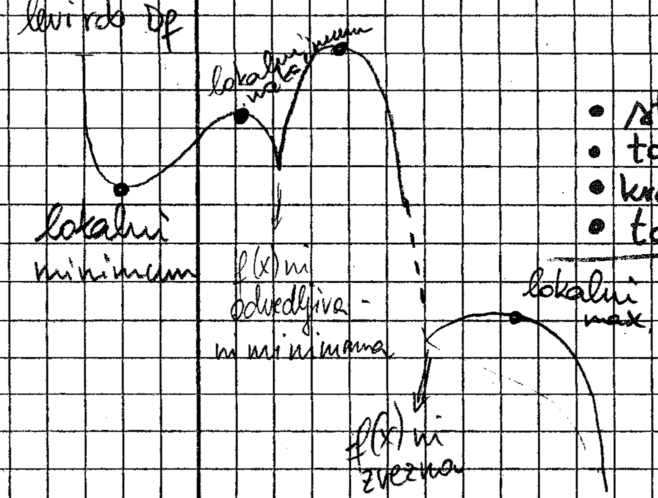
$e^{-x}(x^2-4x+2)$

$f''(0)=1(2)=2 \Rightarrow$ lokalni minimum

$f''(2)=e^{-2}(4-8+2)=e^{-2}\cdot(-2)$

$=-\frac{2}{e^2} \Rightarrow$ lokalni maksimum

Globalni ekstremi



kandidati za globalni ekstrem:

- stacionarne toč
- točke nezveznosti
- krajišča Df oz opazovanega intervala
- točke neodvedljivosti

5. Določi točke kjer f(x) doseže največjo/najmanjšo vrednost na intervalu I:

$f(x) = \frac{x^2-3x}{x+1}$, $I = [0,4]$

kandidati:
- stacionarne točke

$$f'(x) = \frac{(2x-3)(x+1)-(x^2-3x)}{(x+1)^2} = \frac{x^2+2x-3}{(x+1)^2}$$

$$f'(x) = 0$$

$$\frac{x^2+2x-3}{(x+1)^2} = 0 \Rightarrow x^2+2x-3=0$$

$$(x-1)(x+3)=0$$

$\cancel{x_1 = -3}$ ni na intervalu $[0,4]$

$\boxed{x_2 = 1}$

- točke nezveznosti

$f(x)$ ni zv. v polu:

$\cancel{x = -1}$ ni na intervalu $[0,4]$

- krajišča intervala:

$x_1 = 0$
$x_2 = 4$

- točke neodvedljivosti:

$f'(x)$ ni definiran pri $x = -1$

$\cancel{x = -1}$ ni na intervalu $[0,4]$

kandidati:
1, 0, 4

$f(1) = -1 \Rightarrow$ minimum
$f(0) = 0$
$f(4) = \frac{4}{5} \Rightarrow$ maksimum

b) $f(x) = 2\,tg x - tg^2 x$ $\qquad I = [0, \frac{\pi}{2})$

$$f'(x) = 2\frac{1}{\cos^2 x} - 2\,tg x\,\frac{1}{\cos^2 x} = \frac{2}{\cos^2 x}\underbrace{\left(1 - tg x\right)}_{0}$$

$$2\frac{1}{\cos^2 x}\underbrace{(1 - tg x)}_{0} = 0$$

$$tg x = 1$$

$$x = \frac{\pi}{4} + k\pi \qquad k \in \mathbb{Z}$$

- stacionarne točke:

$\frac{\pi}{4}$

- krajišča intervala

$0, \boxed{\frac{\pi}{2}}$ → posebna točka; računamo limite !

- točke nezveznosti:

$\cancel{\frac{\pi}{2} + k\pi}$ ni na intervalu $[0, \frac{\pi}{2})$

- točke neodvedljivosti

ni na intervalu $\left\{ \begin{array}{ll} \cancel{\frac{\pi}{2} + k\pi} & \} \ tg x \text{ ni definiran} \\ \cancel{\frac{\pi}{2} + k\pi} & \} \ \cos x = 0 \end{array} \right.$

kandidati:

$\frac{\pi}{4}, 0, \boxed{\frac{\pi}{2}}$!

$f(\frac{\pi}{4}) = 2\,tg(\frac{\pi}{4}) - (tg(\frac{\pi}{4}))^2 = 2 - 1 = 1$

$f(0) = 0$

$$\lim_{x \uparrow \frac{\pi}{2}} f(x) = \lim_{x \uparrow \frac{\pi}{2}} f(\tfrac{\pi}{2}) = \lim_{x \uparrow \frac{\pi}{2}} (2\underbrace{tg(x)}_{\infty} - \underbrace{tg^2(x)}_{-\infty}) = \lim_{x \uparrow \frac{\pi}{2}} (\underbrace{tg(x)}_{\infty}(2 - \underbrace{tg(x)}_{-\infty})) =$$

$$= -\infty \qquad \infty \cdot (-\infty)$$

funkcija nima minimuma

---

⑥ razstavi št. 36 na produkt dveh faktorjev, tako da bo vsota njunih kvadratov najmanjša.

$36 = x \cdot y \Rightarrow y = \frac{36}{x}$

$x^2 + y^2 = \min$

$x^2 + y^2 = x^2 + \left(\frac{36}{x}\right)^2$

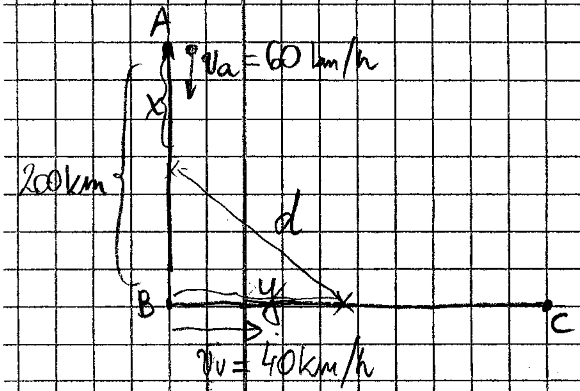
$f(x) = x^2 + \left(\frac{36}{x}\right)^2$

R: $36 = 6 \cdot 6 = (-6)(-6)$

$x_{1,2} = \pm 6$

DN!

⑦ Točke A, B in C ležijo v ravnini, tako da je ABC enak 90°, razdalja med točkama A in B je 200km.
Avto odpelje iz točke A proti točki B z $v = 60 km/h$ v istem trenutku odpelje vlak iz točke B proti točki C z $v = 40 km/h$.
Koliko časa po začetku potovanja bo zr. razdalja med avtom in vlakom najmanjša.



$t = ?$

$x = v_A \cdot t$
$y = v_v \cdot t$

$d = min$

$d = \sqrt{(200-x)^2 + y^2}$
$d^2 = (200-x)^2 + y^2$
} min dosežeta hkrati

$$d^2 = (200 - v_A t)^2 + (v_v \cdot t)^2$$

$$\boxed{f(t) = (200-60t)^2 + (40t)^2} \Longrightarrow \text{iščemo globalni minimum}$$

- točke nezveznosti /
- krajišča D_f: $t \in [0, \infty)$, $\emptyset$
- stac. točke:

$$f'(t) = 2(200-60t)(-60) + 2(40t)40 = -120(200-60t) + 80(40t)$$

$$f'(t) = -120 \cdot 200 + 120 \cdot 60t + 80 \cdot 40t = -24000 + 7200t + 3200t =$$

$$= -24000 + 10400t = 0$$

$$t = \frac{240\cancel{00}}{104\cancel{00}} = \frac{30}{13}$$

- točke neodvedljivosti: /

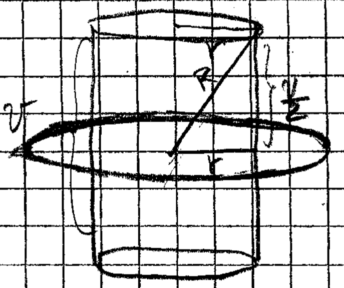
kandidati
$0, \frac{30}{13}$

$$f(0) = (200)^2 = 40000$$

$$f\left(\frac{30}{13}\right) = \left(200 - \frac{60 \cdot 30}{13}\right)^2 + \left(40 \cdot \frac{30}{13}\right)^2 = \left(200 - \frac{180}{13}\right)^2 + \left(\frac{1200}{13}\right)^2$$

$$\boxed{R: t = \frac{30}{13} h = 2h\ 18min}$$

⑧ Krogli z radijem R očrtaj valj z največjim volumnom. Določi r valja



$V = \max$

$V = \pi r^2 \cdot v$
$\quad\; \uparrow \quad \uparrow$

$R^2 = r^2 + \left(\frac{v}{2}\right)^2$

$r^2 = R^2 - \left(\frac{v}{2}\right)^2$

$V = \pi \left(R^2 - \left(\frac{v}{2}\right)^2\right) \cdot v = \pi \underset{\uparrow}{R^2} v - \frac{\pi}{4} v^3$

parameter, ni neznanka

$\boxed{V(v) = \pi R^2 v - \frac{\pi}{4} v^3}$ globalni maksimum

- točke nezveznosti /
- krajišča Df: $\emptyset, 2R \Rightarrow v = \emptyset$
- stacionarne točke

- točke neodvedljivosti /

$V'(v) = \pi R^2 - \frac{\pi}{4} 3v^2$

$v^2 = \frac{4\cancel{\pi}R^2}{3\cancel{\pi}} = \frac{4}{3}R^2$

$v = \sqrt{\frac{4}{3}R^2} \Rightarrow + \frac{2R}{\sqrt{3}} = \boxed{+\frac{2\sqrt{3}R}{3}}$

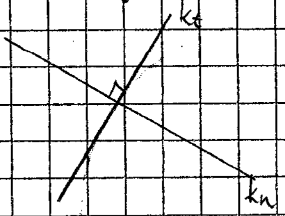
R:

$v = \frac{2\sqrt{3}R}{3}$

$r^2 = R^2 - \left(\frac{v}{2}\right)^2 = R^2 - \frac{R^2}{3} = \frac{2}{3}R^2$

$\boxed{r = \sqrt{\frac{2}{3}}R}$

TANGENTE in NORMALE



$k_t = f'(a)$

$k_n = -\frac{1}{f'(a)}$

⑨ Zapiši enačbi tangente in normale na krivuljo v dani točki T:

a) $y = x^3 + 2x^2 - 4x - 3$
$T(-2, 5)$

$y' = 3x^2 + 4x - 4$

$y'(-2) = 12 - 8 - 4 = 0 \rightarrow$ tangenta je vodorav

$\boxed{y = 5}$ enačba tangente

enačba normale

$\boxed{x = -2}$

DN

b) $y = \arcsin \frac{x-1}{2}$ T_1 = presečišče z x-osjo

R: $y = \frac{1}{2}x - \frac{1}{2}$ tang.

$y = -2x + 2$ norm.

20.12.2010

c) $y = e^{1-x^2}$, T = presečišče s premico $y = 1$

$y = 1$, $1 = e^{1-x^2}$

$\ln 1 = 1 - x^2$

$\underline{x = \pm 1}$

$T_1(1, 1)$

$T_2(1, -1)$

$y' = -2xe^{1-x^2}$

$y'_{1(t)} = -2$; $y(1n) = \frac{1}{2}$ ← (1, 1)

$y'_{2(t)} = +2$; $y'(2n) = -\frac{1}{2}$ ← (−1, 1)

$y_1 = -2x + 3$

$y_2 = 2x + 3$

$y_n = \frac{1}{2}x + \frac{1}{2}$

$y_n = -\frac{1}{2}x + \frac{1}{2}$

② S pomočjo diferenciala izračunaj približno vrednost funkcije:

$f(x) = \sqrt{4x^2 + 5x + 36}$; $x = \frac{9}{100}$

približek z diferencialom:

$$\boxed{f(x) \approx f(a) + f'(a)(x - a)}$$ diferencial

a blizu x df/a

f(a) znamo natančno izračunati

f'(a) — // —

$a = 0$

$f'(x) = \frac{8x + 5}{2\sqrt{4x^2 + 5x + 36}}$

$f(0) = 6$

$f'(0) = \frac{5}{12}$

$f\left(\frac{9}{100}\right) \approx f(0) + f'(0)\left(\frac{9}{100} - 0\right) =$

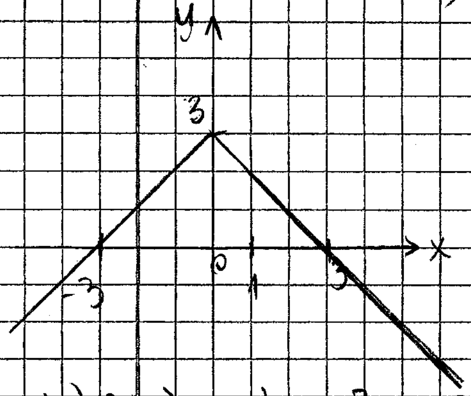
$= 6 + \frac{\cancel{5}^{1}}{\cancel{12}_{4}} \cdot \frac{\cancel{9}^{3}}{\cancel{100}_{20}} = 6 + \frac{3}{80} = \underline{\underline{\frac{483}{80}}}$

GRAFI ~~FUNKCIJ~~ POLINOMOV

③ Nariši grafe odsekoma polinomskih funkcij

a) $f(x) = 3 - |x| = \begin{cases} 3-x; & x \geq 0 \\ 3+x; & x < 0 \end{cases}$

$$\| \\ \begin{cases} x; & x \geq 0 \\ x; & x < 0 \end{cases}$$



b) $f(x) = x^4 - x^3 - 2x^2$

$n: \; x^2(x^2 - x - 2) = 0$

$x^2(x+1)(x-2) = 0$

$x_{1,2} = 0$ … 2. st.

$x_3 = -1$

$x_2 = 2$ } 1. st.

$f(0) = 0 \; ; \; P_y = 0$

$f'(x) = 4x^3 - 3x^2 - 4x$

$x(4x^2 - 3x - 4) = 0$

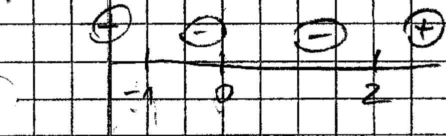
$x_1 = 0$

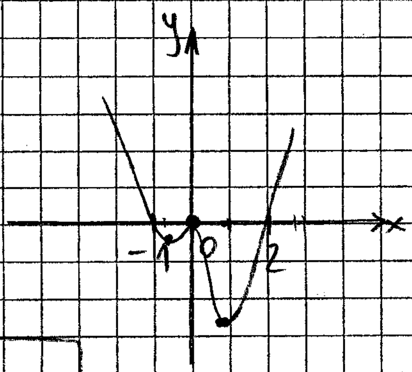
$x_{2,3} = \frac{3 \pm \sqrt{9+64}}{8} = \frac{3 \pm \sqrt{73}}{8}$

$x_2 \doteq 1{,}44 \doteq \underline{1{,}4}$ $\quad y_2 = \underline{-2{,}2}$ lok. min.

$x_3 \doteq -0{,}68 \doteq \underline{-0{,}7}$ $\quad y_3 = \underline{-0{,}4}$ lok. min.

predznak



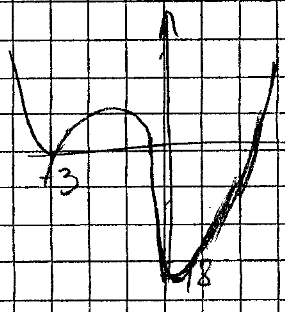


c) $f(x) = 2x^4 + 9x^3 - 2x^2 - 39x - 18$

HORNERJEV ALGORITEM

DN

R:



④

GRAFI RACIONALNIH FUNKCIJ

a) $f(x)=\frac{2x^2-4x}{(1-x^2)(x-2)}=\frac{2x\cancel{(x-2)}}{(1-x^2)\cancel{(x-2)}}$

nujno krajšati!

sicer ~~bi~~ bila v isti točki ničla in pol!

ničle
poli
asimptota
f(0)
~~stacionarne točke~~
predznak

ničle: $2x=0$

$x=0$

poli: $1-x^2=0$

$x=\pm 1$

asimptota: $2x : x^2-1 \Rightarrow \underline{a: y=0}$

$\boxed{st.\ im. > st.\ št \Rightarrow a:\ y=0}$

$f(0)=0$

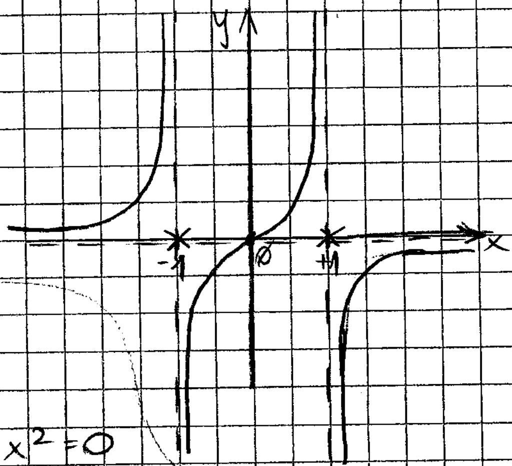
$f'(x)=\frac{2(1-x^2)-2x(-2x)}{(1-x^2)^2}=\frac{2-2x^2+4x^2}{(1-x^2)^2}=\frac{2+2x^2}{(1-x^2)^2}=\frac{2(1+x^2)}{(1-x^2)^2}$

$2(1+x^2)=0$

nima realne rešitve

predznak ⊕ ⊕ ⊕ $f(2)=-\frac{4}{3}$

⊕ -1 ⊖ 0 ⊕ 1 ⊖



b) $f(x)=\frac{4(x^2-x^4)}{1-4x^2}$

n: $4(x^2-x^4)=\cancel{4}4(x^2(1-x^2)=0$

$x_{1,2}=0$

$x_3=1$

$x_4=-1$

p: $1-4x^2=0$

$(1-2x)(1+2x)$

$x_1=-\frac{1}{2}$

$x_2=\frac{1}{2}$

$f'(x)=\frac{(8x-16x^3)(1-4x^2)-(4x^2-4x^4)(8x)}{(1-4x^2)^2}$

$(4x^2-4x^4):(1-4x^2)=x^2-\frac{3}{4}$

$-x^2+4x^4$

$3x^2$

$\frac{3}{4}-3x^2$

$\frac{3}{4}$

a: $y=x^2-\frac{3}{4}$

$f'(x)=\frac{8x-\cancel{32x^3}-16x^3+64x^5-(\cancel{32x^3}+32x^5)}{(1-4x^2)^2}$

$x_{1,2}=\frac{0\pm\sqrt{0+3}}{2}=\pm\frac{\sqrt{3}}{2}$

$f'(x)=\frac{32x^5-16x^3+8x}{(1-4x^2)^2}=$

$=\frac{8x(4x^4-2x^2+1)}{(1-4x^2)^2}$

$4x^4-2x^2+1$ $D<0$

nima realnih ničel

+ -1 - + 0 + - 1 +

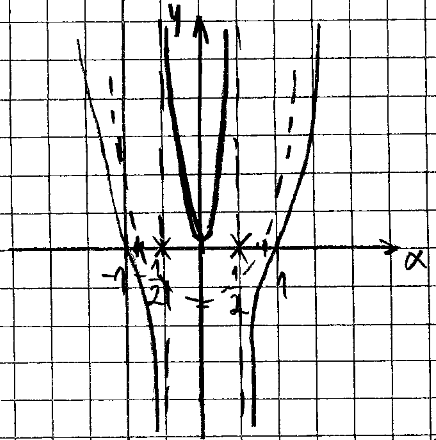
$a' \; y = x^2 - \frac{3}{4}$

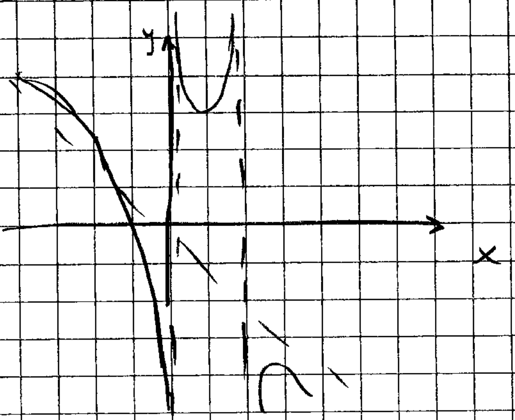
$2x' = 0$

$x' = 0$

$T(0, -\frac{3}{4})$

$x_{1,2} = \pm\frac{\sqrt{3}}{2}$





DN
c) $f(x) = \frac{1+x^3}{x-x^2}$ R:

5. nariši grafe

a) $f(x) = x^3 e^{-x}$

ničle
Df
limita: $\lim\limits_{x\to\infty} f(x)$, $\lim\limits_{x\to-\infty} f(x)$
začetno vrednost
stacionarne točke
intervali naraščanja in padanja

- ničle:

$f(x) = 0$

$x^3 e^{-x} = 0$

$x = 0$; III. stopnje

- $D_f = \mathbb{R}$

$\to \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} x^3 e^{-x} = \lim\limits_{x\to\infty} \frac{x^3}{e^x} = \boxed{0}$ ✓

$\infty \cdot 0$?

ker e^x narašča hitreje k

x^n

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \underset{-\infty}{\underset{\downarrow}{x^3}} \, \underset{\infty}{\underset{\downarrow}{e^{-x}}} = -\infty$

- $f(0) = 0$

- stacionarne točke:

$f'(x) = 3x^2 e^{-x} - x^3 e^{-x} = x^2 e^{-x}(3-x) = 0$

$x_1 = 0 \quad y_1 = 0$
$x_2 = 3 \quad y_2 = 27/e^3 > 1$

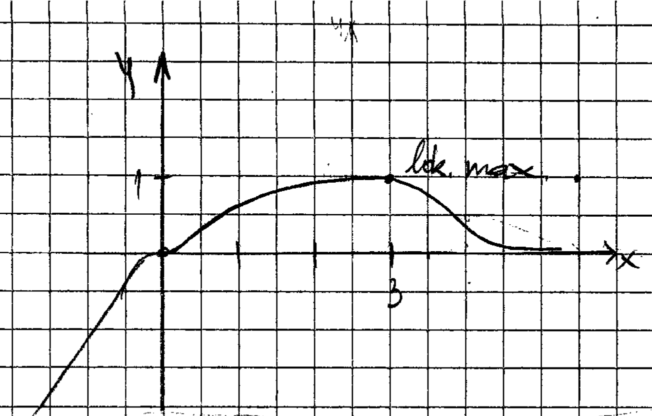
$\Downarrow$

$\doteq 1{,}3$

intervali:

$f'(x) > 0$, narašča ; $3 - x > 0$; $\underline{x < 3}$

pada $x > 3$



$\boxed{\text{DN b)}\quad f(x)=e^{-x^2}}$

c) $f(x) = 2\sin\left(\frac{x}{2} - \frac{2\pi}{3}\right) = 2\sin\left(\frac{1}{2}\left(x - \frac{4\pi}{3}\right)\right)$

$\frac{1}{2}$ manjša frekvenca

premik v desno

za $\frac{4\pi}{3}$

2. način:

$\boxed{\text{ničle}}$ $\frac{x}{2}\ \frac{2\pi}{3} = 0 + k\pi$, $k \in \mathbb{Z}$ $\quad -\frac{8\pi}{3}, -\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{10\pi}{3}, \frac{16\pi}{3}$

$x = \frac{4\pi}{3} + 2k\pi$

$\boxed{\text{lok. max}}$ $\frac{x}{2} - \frac{2\pi}{3} = \frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$

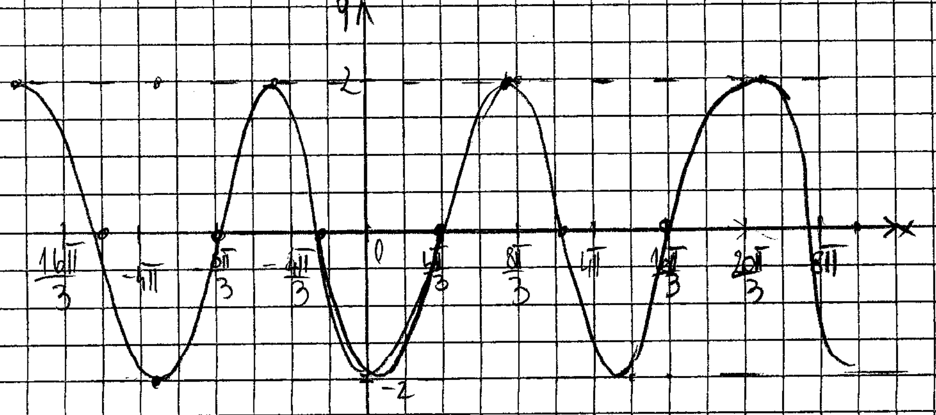
$\frac{x}{2} = \frac{2\pi}{3} + \frac{\pi}{2} + 2k\pi$ $\quad -\frac{17}{3}\pi, -\frac{5\pi}{3}, \frac{7\pi}{3}, \frac{19\pi}{3}$

$x = \frac{7\pi}{3} + 4k\pi$

$\boxed{\text{lok. min}}$ $\frac{x}{2} - \frac{2\pi}{2} = -\frac{\pi}{2} + 2k\pi$ $\quad k \in \mathbb{Z}$

$\frac{x}{2} = \frac{2\pi}{3} - \frac{\pi}{2} + 2k\pi$ $\quad -\frac{11\pi}{3}, \frac{\pi}{3}, \frac{13\pi}{3}$

$x = \frac{\pi}{3} + 4k\pi$



INTEGRAL

- nedoločeni integral:

PRAVILA

$$\int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$$

$$\int c \cdot f(x) dx = c \int f(x)$$

PER PARTES

$$\int u dv = uv - \int v du$$

NOVA SPREMENLJIVKA

$$\int f(x) dx = \int f(g(t)) \cdot g'(t) dt$$

$$x = g(t) \Longrightarrow 1 dx = g'(t) dt$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + c \Longleftarrow \int \frac{dx}{1+x^2} = \operatorname{arctg} x + c$$

x $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + c$$

x $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2} + c$

x $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right) + c$

---

a) $\int \left(1 - \frac{1}{x^2}\right) \sqrt{x\sqrt{x}}\, dx = \int (1 - x^{-2}) x^{1/2} \cdot x^{1/4} dx = \int (1 - x^{-2}) x^{3/4} dx =$

$= \int \left(x^{3/4} - x^{-5/4}\right) dx = \frac{4}{7} x^{7/4} + 4 x^{-\frac{1}{4}} + c$

b) $\int \frac{x^4}{1+x^2} dx = \int \left(x^2 - 1 + \frac{1}{1+x^2}\right) dx = \frac{x^3}{3} - x + \arctan x + c$

$$x^4 : 1 + x^2 = x^2 - 1$$

$$\underline{x^2 \quad + x^4}$$

$$\underline{\quad -x^2}$$

$$\underline{-1 - x^2}$$

$$1 \text{ ost.}$$

23. 12. 201

① Nariši graf funkcije:

a) $f(x) = \sqrt[3]{x^2 + x^3} = (x^2 + x^3)^{1/3}$

- ničle:

$\sqrt[3]{x^2 + x^3} = 0$

$x^2 + x^3 = 0$

$x^2(1 + x) = 0$

$x_{1,2} = 0$

$x_3 = -1$

- D_f:

$x \in \mathbb{R}$

- $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \sqrt[3]{x^2 + x^3} = \underline{\underline{\infty}}$

- $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to \infty} \sqrt[3]{\underset{\underset{-\infty}{\Downarrow}}{x^2} + \underset{\underset{\infty}{\Downarrow}}{x^3}} =$ nedoločeno

$= \lim\limits_{x \to -\infty} \sqrt[3]{\underset{\underset{\infty}{\downarrow}}{x^2}\underset{\underset{(-\infty)}{\downarrow}}{(1 + x)}} = \underline{\underline{-\infty}}$

- stacionarne točke

$$f'(x) = \frac{1}{3}(x^2 + x^3)^{-\frac{2}{3}} \cdot (2x + 3x^2) = \frac{2x + 3x^2}{3\sqrt[3]{(x^2 + x^3)^2}}$$

$f'(x) = 0$

$2x + 3x^2 = 0$

$x(2 + 3x) = 0$

$x_1 = 0, \ x_2 = -\frac{2}{3}$

$f(x_1) = 0$

$f(x_2) = \sqrt[3]{\frac{4}{9} - \frac{8}{27}} = \sqrt[3]{\frac{4}{27}} = \frac{\sqrt[3]{4}}{3}$

- intervali naraščanja/padanja

$f'(x) > 0$

$\frac{2x + 3x^2}{3\sqrt[3]{(x^2 + x^3)^2}} > 0$

množimo lahko, ker je spodnji izraz $\underline{\underline{{}^{\wedge}2}}$

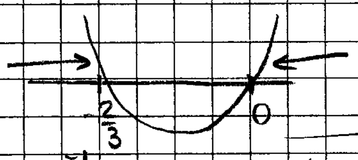
$2x + 3x^2 > 0$

$x(2 + 3x) > 0$

$x_1 = 0$

$x_2 = -\frac{2}{3}$

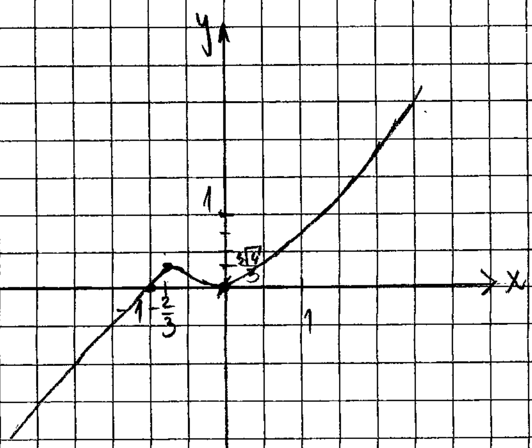
kvadratna neenačba



$x \in (-\infty, -\frac{2}{3}) \cup (0, \infty)$

narašča

$x \in (-\frac{2}{3}, 0)$ pada



c) $f(x) = 2\sin\left(\frac{x}{2} - \frac{2\pi}{3}\right) = 2\sin\left(\frac{1}{2}\left(x - \frac{4\pi}{3}\right)\right)$

amplituda!
razteg v smeri y osi za faktor 2

frekvenca!
razteg v smeri osi x za faktor 2

faza!
premik v desno za $\frac{4\pi}{3}$

- ničle: $\frac{x}{2} - \frac{2\pi}{3} = 0 + k\pi \qquad k \in \mathbb{Z}$

$x = \frac{4\pi}{3} + 2k\pi \qquad \ldots, -\frac{8\pi}{3}, -\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{10\pi}{3}, \frac{16\pi}{3}, \ldots$

- lok. max. $\frac{x}{2} - \frac{2\pi}{3} = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$

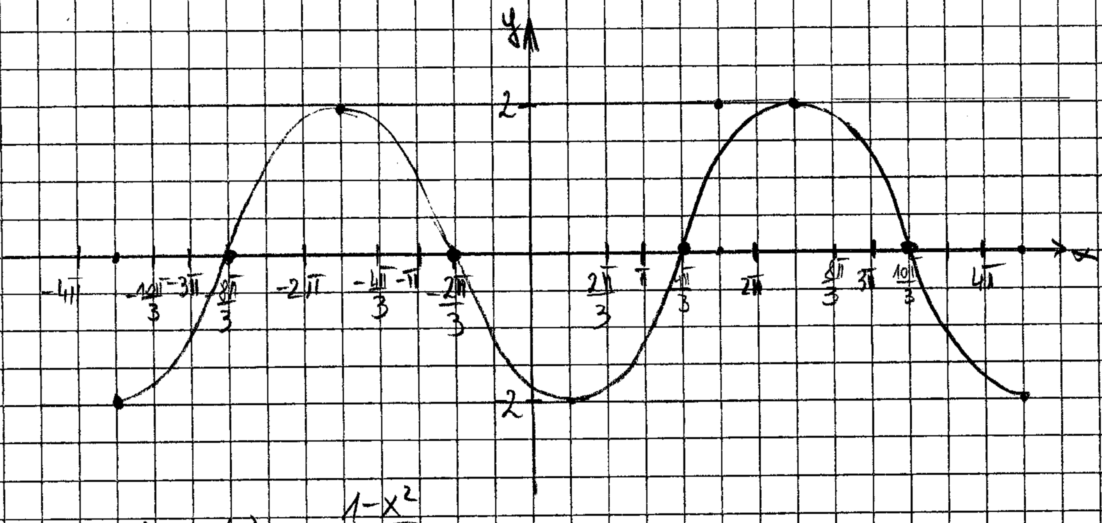
$\frac{x}{2} = \frac{2\pi}{3} + \frac{\pi}{2} + 2k\pi \Rightarrow x = \frac{7\pi}{3} + 4k\pi \qquad \ldots, -\frac{17\pi}{3}, -\frac{5\pi}{3}, \frac{7\pi}{3}, \frac{19\pi}{3}, \ldots$

- lok. min. $\frac{x}{2} - \frac{2\pi}{3} = -\frac{\pi}{2} + 2k\pi, \; k \in \mathbb{Z}$

$\ldots, -\frac{11\pi}{3}, \frac{\pi}{3}, \frac{13\pi}{3}, \ldots$

$\frac{x}{2} = \frac{2\pi}{3} - \frac{\pi}{2} + 2k\pi$

$x = \frac{\pi}{3} + 4k\pi$



2. Dana je $f(x) = \frac{1-x^2}{1+x^2}$

a) poišči ničli, asimptoto in ekstreme ter nariši graf $f(x)$

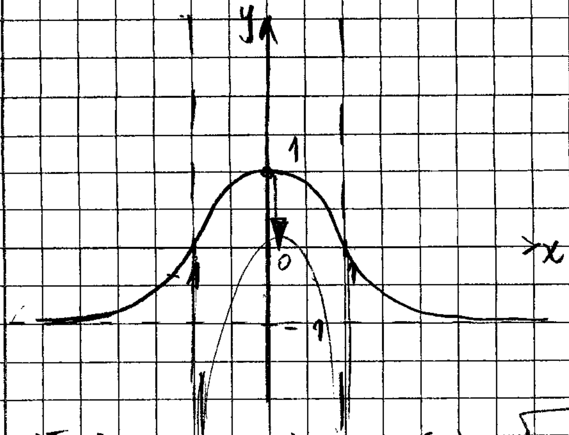
$f(x) = \frac{1-x^2}{1+x^2}$

a: $y = -1$

$f'(x) = \frac{-2x(1+x^2) - (1-x^2)2x}{(1+x^2)^2} = \frac{\cancel{2x^3} - 2x - 2x + \cancel{2x^3}}{(1+x^2)^2} = \frac{-4x}{(1+x^2}$

ničle: $x_1 = 1$
$x_2 = -1$

$-4x = 0$
$x = 0$



$g(x) = \ln(f(x))$

$D_g = \{x;\ f(x) > 0\} = (-1,1)$ iz grafa

$Z_g\ (-\infty, 0]$

b) poišči D_f in Z_f $f(x) := g(x) = \ln(f(x))$

NEDOLOČENI INTEGRAL

Pravila: $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$$\int c f(x)\,dx = c \int f(x)\,dx$$

per partes: $\int u\,dv = uv - \int v\,du$

nova spremenljivka

$$\int f(x)\,dx = \int f(g(t)) \cdot g'(t)\,dt \qquad x = g(t)$$

$$dx = g'(t)\,dt$$

$\int dx = x + c$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c\ ;\ n \neq 1$

$\int \frac{1}{x}\,dx = \ln|x| + c$

$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + c$

$\int \sin x\,dx = -\cos x + c$

$\int \cos x\,dx = \sin x + c$

$\int e^x\,dx = e^x + c$

$\int a^x\,dx = \frac{a^x}{\ln x} + c$

$\int \frac{dx}{\cos^2 x}\,dx = \operatorname{tg} x + c$

$\int \frac{dx}{\sin^2 x}\,dx = -\operatorname{ctg} x$

$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + c$

a) $\int \left(1-\frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\,dx = \int (1-x^{-2})\,x^{3/4}dx = \int \left(x^{3/4}-x^{-\frac{5}{4}}\right)dx =$

$= \frac{x^{7/4}}{\frac{7}{4}} - \frac{x^{-1/4}}{-\frac{1}{4}} + c = \frac{4}{7}x^{7/4} + 4x^{-1/4} + c$

b) $\int \frac{x^4}{1+x^2}dx = \int \left(x^2-1+\frac{1}{1+x^2}\right)dx =$

$x^4 : (1+x^2) = x^2-1$
$-(x^2+x^4)$
$-x^2$
$+1+x^2$
1 ost

$= \frac{x^3}{3} - x + \operatorname{arctg}x + c$

c)

$\int tg^2x\,dx = \int \frac{\sin^2x}{\cos^2x}dx = \int \frac{1-\cos^2x}{\cos^2x}dx = \int\left(\frac{1}{\cos^2x}-1\right)dx =$

$tgx - x + c$

4. Izračunaj z uporabo nove spremenljivke

a) $\int \sqrt[3]{1-3x}\,dx = \int (1-3x)^{1/3}dx = -\frac{1}{3}\int t^{1/3}dt =$

$t = 1-3x$
$dt = -3dx$
$dx = -\frac{dt}{3}$

$= -\frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{4}t^{4/3} = -\frac{1}{4}(1-3x)^{4/3}+c$

b) I. $\int \frac{x}{3x+2}dx = \int \frac{1}{3} - \frac{2}{9}\frac{1}{(x+\frac{2}{3})}dx =$

$x : (3x+2) = \frac{1}{3} - \frac{\frac{2}{3}}{3x+2}$
$x + \frac{2}{3}$
$-\frac{2}{3}$

$= \frac{1}{3}x - \frac{2}{9}\ln\left|x+\frac{2}{3}\right| + c$

$\frac{1}{3} - \frac{2}{9}\frac{1}{(x+\frac{2}{3})}$

II. $t = 3x+2 \Rightarrow x = \frac{t-2}{3}$
$dt = 3dx \Rightarrow dx = \frac{dt}{3}$

$= \int \frac{\frac{t-2}{3}}{t}\cdot\frac{dt}{3} = \frac{1}{9}\int\frac{t-2}{t}dt = \frac{1}{9}\int\left(1-\frac{2}{t}\right)dt = \frac{1}{9}\left(t-2\ln|t|\right)+c =$

$= \frac{1}{9}(3x+2) - \frac{2}{9}\ln(3x+2)+c$

$\int \frac{1}{x+a}dx = \ln|x+a|+c$

$\int e^{ax}dx = \frac{1}{a}e^{ax}+c$

$\int \sin(ax)dx = -\frac{1}{a}\cos(ax)+c$

$\int \cos x(ax)dx = \frac{1}{a}\sin(ax)+c$

d) $\int tg x \, dx = \int \frac{\sin x}{\cos x} dx = \int -\frac{dt}{t} = -\ln|t| + c = -\ln|\cos x| + c$

$t = \cos x$

$dt = -\sin x \, dx \Rightarrow \sin x \, dx = -dt$

$dx = -\frac{dt}{\sin x}$

e) $\int \frac{\sin x \cdot \cos^2 x}{1+\cos^2 x} dx = -\int \frac{\cancel{\sin x}}{\cancel{\sin x}} \frac{t^2}{1+t^2} dt = -\int \left(1 - \frac{1}{1+t^2}\right) dt =$

$t = \cos x$

$dt = -\sin x \, dx$

$dx = \frac{dt}{-\sin x}$

$t^2 : (1+t^2) = 1$

$\underline{-(1+t^2)}$

ost (-1)

$= -t + \text{arctg}\, t + c =$

$= -\cos x + \text{arctg} \cos x + c$

f) D.N.

$\int e^{\sin x} \cos x \, dx$

R: $e^{\sin x} + c$

h) DN

$\int \frac{\ln^2 x}{x} dx$

R: $\frac{\ln^3 x}{3} + c$

g(x) $\int \frac{dx}{x \ln x \ln(\ln x)} = \int \frac{du}{u} = \ln|u| + c = \ln|\ln(\ln x)| + c$

~~$t = \ln x$~~

$u = \ln(\ln x)$

$du = \frac{1}{\ln x} \cdot \frac{1}{x} dx$

$\boxed{x^3 = x^2 \cdot x}$

i) $\int x^3 \sqrt{1-x^2} dx = \int x^3 (1-x^2)^{1/2} dx = \int x^2 (1-x^2)^{1/2} x \, dx =$

$t = 1 - x^2$

$dt = -2x \, dx$

$x \, dx = -\frac{dt}{2}$

$= \int (1-t) t^{1/2} \frac{dt}{(-2)} = -\frac{1}{2} \int (t^{1/2} - t^{3/2}) dt =$

$= -\frac{1}{2} \left(\frac{2}{3} t^{3/2} - \frac{2}{5} t^{5/2}\right) + c = -\frac{1}{3} t^{3/2} + \frac{1}{5} t^{5/2} + c =$

$= -\frac{(1+x^2)^{3/2}}{3} + \frac{(1-x^2)^{5/2}}{5} + c$

g) DN. $\int \sin^2 x\,dx$

R: $\frac{1}{2}x - \frac{1}{4}\sin 2x + c$

$\sin^2 x = \frac{1-\cos(2x)}{2}$

$\cos^2 x = \frac{1+\cos(2x)}{2}$

PER PARTES

a) $\int x^2 e^x dx = e^x x^2 - \int e^x \cdot 2x\,dx = e^x x^2 - \left(2xe^x - \int e^x 2dx\right) =$

$u = x^2 \quad dv = e^x dx$

$du = 2x dx \quad v = e^x$

mora se lepo odvajati

more se lepo integrirati

$= e^x x^2 - \left(2xe^x - 2e^x\right) + c$

$= e^x x^2 - 2e^x\left(x-1\right) + c$

$u = 2x \quad dv = e^x dx$

$du = 2dx \quad v = e^x$

$= e^x\left(x^2 - 2x + 2\right) + c$

b) $\int \arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx =$

$u = \arcsin x \quad du = \frac{1}{\sqrt{1-x^2}}dx$

$dv = dx \quad v = 1x$

$t = 1-x^2$

$dt = -2x dx$

$= x\arcsin x - \int t^{\frac{1}{2}} \frac{dt}{-2} =$

$= x\arcsin x + \frac{1}{\cancel{2}} \cdot \cancel{2} \cdot t^{1/2} + C =$

$= x\arcsin x + \sqrt{1-x^2} + c$

c) DN $\int x\ln(x-1)\,dx$

R: $\frac{1}{2}x^2\ln(x-1) - \frac{1}{4}x^2 - \frac{1}{2}x\ln(x-1) + c$

d) $\overset{I}{\int e^{ax}\cos(bx)\,dx}$; $a, b \in R$

$u = e^{ax} \Rightarrow du = \frac{1}{a}e^{ax}dx$

$dv = \cos(bx)dx \Rightarrow v = \frac{1}{b}\sin(bx)$

$e^{ax}\frac{1}{b}\sin(bx) - \int \frac{a}{b}\sin(bx)\cdot e^{ax}dx$

$u = e^{ax} \quad du = ae^{ax}dx$

$dv = \sin(bx)dx \quad v = \frac{1}{b}\cos(bx)$

$= e^{ax}\frac{1}{b}\sin(bx) - \frac{a}{b}\left(e^{ax}\frac{1}{b}\cos(bx) + \int \frac{a}{b}\underbrace{e^{ax}\cos(bx)dx}_{I}\right)$

$I = \frac{1}{b}e^{ax}\sin(bx) + \frac{a}{b^2}e^{ax}\cos(bx) - \frac{a^2}{b^2}I$

$$I\left(1+\frac{a^2}{b^2}\right)=\frac{1}{b}e^{ax}\sin(bx)+\frac{a}{b^2}e^{ax}\cos(bx)$$

$$I=\frac{\frac{1}{b^2}e^{ax}\left(b\sin(bx)+a\cos(bx)\right)}{1+\frac{a^2}{b^2}}$$

3. 1. 2011

$$\int tg^2x\,dx=\int\frac{\sin^2x}{\cos^2x}dx=\int\frac{1-\cos^2x}{\cos^2x}dx=\int\left(\frac{1}{\cos^2x}-1\right)dx=tgx-x+c$$

$$\int\sin^2x\,dx=\int\frac{1-\cos2x}{2}dx=\frac{1}{2}\int(1-\cos(2x))dx=$$

$$\frac{1}{2}x-\frac{1}{4}\sin2x+c$$

$$\sin^2x=\frac{1-\cos(2x)}{2}$$

$$\cos^2x=\frac{1+\cos(2x)}{2}$$

$$\int\sin(ax)dx=-\frac{1}{a}\cos(ax$$

$$\int\cos(ax)dx=\frac{1}{a}\sin(ax)+c$$

$$\int e^{ax}dx=\frac{1}{a}e^{ax}+c$$

$$\int\frac{dx}{x+a}=\ln|x+a|$$

② nova spremenljivka

a) $\int\sqrt[3]{1-3x}\,dx=\int(1-3x)^{1/3}dx=-\frac{1}{3}\int t^{1/3}dt=$

$1-3x=t$
$-3dx=dt$
$dx=-\frac{dt}{3}$

$$=-\frac{1}{\cancel{3}}\frac{\cancel{3}}{4}t^{4/3}+c=-\frac{1}{4}\sqrt[3]{(1-3x)^4}+c$$

b) $\int\frac{x}{3x+2}dx=\int\frac{t-2}{3t}\cdot\frac{dt}{3}=$

$3x+2=t \Rightarrow x=\frac{t-2}{3}$
$3dx=dt$
$dx=\frac{dt}{3}$

$$=\frac{1}{9}\int\frac{t-2}{t}dt=\frac{1}{9}\int\left(1-\frac{2}{t}\right)dt=$$

$$\frac{1}{9}t-\frac{1}{9}2\ln|t|+c=\frac{1}{9}(3x+2)-2\ln|3x+2|+c$$

c) $\int\boxed{\tan x\,dx}=\int\frac{\sin x}{\cos x}dx$

I. $t=\cos x$, $dt=-\sin x\,dx$ II. $t=\sin x$, $dt=\cos x\,dx$ } bolj komplicirano

I. $\int\frac{\cancel{\sin x}}{t}\frac{dt}{(-\cancel{\sin x})}=-\int\frac{dt}{t}=\boxed{-\ln|\cos x|+c}$

d) $\int\frac{e^x}{1+e^{2x}}dx=\int\frac{dt}{1+t^2}=\arctan t+c=\arctan(e^x)+c$

~~$t=1+e^{2x}$~~
~~$dt=2e^{2x}dx$~~ $t=e^x$ $dt=e^xdx$

e) $\int \frac{\ln^2 x}{x} dx = \int t^2 dt = \frac{t^3}{3} + c = \frac{\ln^3 x}{3} + c$

$t = \ln x$
$dt = \frac{1}{x} dx$

f) $\int e^{\sin x} \cos x \, dx$ R: $e^{\sin x} + c$

e) $\int \frac{dx}{x \ln x \ln(\ln x)} = \int \frac{dt}{t} = \ln|t| + c = \ln|\ln(\ln x)| + c$

$t = \ln(\ln x)$
$dt = \frac{1}{\ln x} \cdot \frac{1}{x} dx$

h) $\int \frac{\operatorname{arctg} \sqrt{x}}{\sqrt{x}(1+x)} dx$ R: $\operatorname{arctg}^2 \sqrt{x} + c$

i)

$$\int x^3 \sqrt{1-x^2} \, dx = \int x \, x^2 \sqrt{1-x^2} \, dx = \int \cancel{x}(1-t)\sqrt{t} \frac{dt}{-2\cancel{x}} =$$

$1 - x^2 = t \Rightarrow x^2 = 1 - t$
$-2x\,dx = dt$
$dx = \frac{dt}{-2x}$

$$= -\frac{1}{2} \int (1-t) t^{1/2} dt =$$
$$= -\frac{1}{2} \int (t^{1/2} - t^{3/2}) dt =$$
$$= -\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} t^{3/2} - \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{5} t^{5/2} + c = -\frac{1}{3}\sqrt{(1-x^2)^3} + \frac{1}{5}\sqrt{(1-x^2)^5} + c$$

PER PARTES

a) $\int x^2 e^x dx$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

$u = x^2$ $du = 2x\,dx$
$dv = e^x dx$ $v = e^x$

$$= e^x x^2 - \int e^x 2x\,dx = e^x x^2 - 2\int e^x x\,dx$$

$u = x$ $du = 1\,dx$
$dv = e^x dx$ $v = e^x$

$$= e^x x^2 - 2\left(e^x \cdot x - \int e^x dx\right) =$$
$$= e^x x^2 - 2xe^x + 2e^x + c$$
$$e^x(x^2 - 2x + 2) + c$$

b) $\int \arcsin x \, dx = \qquad = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx =$

$u = \arcsin x \qquad du = \frac{1}{\sqrt{1-x^2}}dx$

$dv = dx \qquad v = x$

$= x\arcsin x - \int \frac{1}{-2\sqrt{t}}dt =$

$1-x^2 = t$

$-2x\,dx = dt$

$= x\arcsin x + \frac{1}{\cancel{2}}\cancel{2}t^{1/2} + c$

$$\underline{= x\arcsin x + \sqrt{1-x^2} + c}$$

c) $\int x\ln(x-1)\,dx \qquad = \frac{x^2}{2}\cdot\ln(x-1) - \frac{1}{2}\int\frac{x^2}{(x-1)}dx =$

$u = \ln(x-1) \qquad du = \frac{1}{(x-1)}dx$

$dv = x\,dx \qquad v = \frac{x^2}{2}$

$x^2 : x-1 = x+1$

$-x^2 \underset{+}{-} x$

$x \underset{+}{-} 1$

1

$= \frac{x^2}{2}\ln(x-1) - \frac{1}{2}\int\left(x+1 + \frac{1}{x-1}\right)dx =$

$= \frac{x^2}{2}\ln(x-1) - \frac{1}{2}\left(x^2 + x + \ln(x-1)\right) + c$

d) $\int e^{ax}\cos(bx)\,dx \quad ; \quad a, b \in R$

$u = e^{ax} \qquad du = ae^{ax}dx$

$dv = \cos(bx)\,dx \qquad v = \frac{1}{b}\sin(bx)$

$= e^{ax}\cdot\frac{1}{b}\sin(bx) - \int\frac{a}{b}e^{ax}\sin(bx)\,dx$

$u = e^{ax} \qquad du = ae^{ax}dx$

$dv = \frac{a}{b}\sin(bx)\,dx \qquad v = -\frac{a}{b^2}\cos(bx)$

$= \frac{1}{b}e^{ax}\sin(bx) + \frac{a}{b^2}\cos(bx)e^{ax} - \int\frac{a^2}{b^2}e^{ax}\cos(bx)\,dx$

$I = \frac{1}{b^2}e^{ax}\left(b\sin(bx) + a\cos(bx)\right) - \frac{a^2}{b^2}\cdot I$

$\left(1 + \frac{a^2}{b^2}\right)I = \frac{1}{b^2}e^{ax}\left(b\sin(bx) + a\cos(bx)\right)$

$$I = \frac{\cancel{\frac{1}{b^2}}e^{ax}\left(b\sin(bx) + a\cos(bx)\right)}{\cancel{\frac{1}{b^2}}\left(b^2 + a^2\right)}$$

e) $\int \operatorname{arctg}\sqrt{x}\, dx$

R: $x \operatorname{arctg}\sqrt{x} - \sqrt{x} + \operatorname{arctg}\sqrt{x} + C$

Integral racionalne funkcije

$\int \frac{p(x)}{q(x)} dx$, $st(p) \geq st(q) \Rightarrow$ **DELIMO!**

a) $\int \frac{x}{(x+1)(x+2)(x+3)} dx$ } razdelimo na parcialne ulomke

$$\frac{x}{(x+1)(x+2)(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+3)} =$$

$$\frac{x}{(x+1)(x+2)(x+3)} = \frac{A(x+3)(x+2) + B(x+1)(x+3) + C(x+2)(x+1)}{(x+1)(x+2)(x+3)}$$

$$x = A(x^2+5x+6) + B(x^2+4x+3) + C(x^2+3x+2)$$

x^2: $0 = A+B+C$

x^1: $1 = 5A+4B+3C$ $\Big) (-3)$

x^0: $0 = 6A+3B+2C$ $\Big) (-2)$

$-1 = -2A - B$

$0 = -4A - B$ $\Big) -$

$-1 = 2A$

$A = -\frac{1}{2}$

$B = 2$

$C = -\frac{3}{2}$

$$= \int \left(\frac{-\frac{1}{2}}{x+1} + \frac{2}{x+2} + \frac{-\frac{3}{2}}{x+3} \right) dx$$

$$= -\frac{1}{2} \ln|x+1| + 2\ln|x+2| - \frac{3}{2}\ln|x+3| + C$$

$$\int \frac{dx}{x^3+x^2+2x+2} \Rightarrow \frac{1}{x^2(x+1)+2(x+1)} = \frac{1}{(x+1)\underbrace{(x^2+2)}_{\text{nerazcepen faktor}}} =$$

$$= \frac{1}{(x+1)(x^2+2)} = \frac{Bx+C}{(x^2+2)} + \frac{A}{x+1}$$

$$\frac{1}{(x+1)(x^2+2)} = \frac{(Bx+C)(x+1) + A(x^2+2)}{(x^2+2)(x+1)}$$

x^2: $0 = A+B$

x^1: $0 = B+C$

x^0: $1 = 2A+C$

$A+B$

$B+1-2A$

$2A+2B$

$B+1-2A$

$3B+1=0$

$$\boxed{\begin{aligned} B &= -\frac{1}{3} \\ A &= \frac{1}{3} = C \end{aligned}}$$

$$\int \frac{dx}{x^3+x^2+2x+2} = \int\left(\frac{A}{x+1} + \frac{Bx+C}{x^2+2}\right)dx =$$

$$\int\left(\frac{1}{3(x+1)} + \frac{-\frac{x}{3}+\frac{1}{3}}{x^2+2}\right)dx = \frac{1}{3}\ln|x+1| + \int\left(\frac{-\frac{x}{3}}{x^2+2} + \frac{\frac{1}{3}}{x^2+2}\right)dx$$

$$= \frac{1}{3}\ln|x+1| + \frac{1}{3\sqrt{2}}\,\text{arctg}\frac{x}{\sqrt{2}} + \int\frac{\left(-\frac{x}{3}\right)}{x^2+2}dx$$

$$\boxed{\int\frac{dx}{x^2+a^2} = \frac{1}{a}\text{atg}\frac{x}{a}}$$

$$t = x^2+2$$
$$dt = 2x\,dx$$

$$-\frac{1}{3}\int\frac{1}{2t}dt = -\frac{1}{6}\ln|t| = -\frac{1}{6}\ln|x^2+2| + C$$

$$\frac{1}{3}\ln|x+1| + \frac{\sqrt{2}}{6}\,\text{arctg}\frac{x}{\sqrt{2}} - \frac{1}{6}\ln|x^2+2| + C$$

$$\int\frac{1}{x+a}dx \Rightarrow \ln|x+a|$$

$$\int\frac{1}{x^2+ax+b}dx \Rightarrow \ln|x^2+ax+b| + \text{arctg}\frac{\boxed{2x+a}}{\boxed{\sqrt{-D}}}$$

odvod polinoma v imenovalcu

diskriminanta polinoma v imenovalcu

$$\int\frac{x+2}{x^3-2x^2}dx$$

$$\frac{x+2}{x^2(x-2)} = \frac{Ax+B}{x^2} + \frac{C}{x-2} = \frac{(Ax+B)(x-2)+Cx^2}{x^2(x-2)}$$

$$x+2 = Ax^2-2Ax+Bx-2B+Cx^2$$

x^2 : $0 = A + C$

x^1 : $1 = -2A + B$

x^0 : $2 = -2B \Rightarrow \underline{\underline{B=-1}}$

$-2A = 2$

$\underline{\underline{A=-1}}$

$\underline{\underline{C=1}}$

$$\int\frac{x+2}{x^3-2x^2}dx = \int\left(\frac{-x-1}{x^2} + \frac{1}{x-2}\right)dx = \ln|x-2| - \ln|x| + x^{-1} + C =$$

$$= \underline{\underline{\boxed{\frac{1}{x}} + \ln\frac{|x-2|}{|x|} + C}}$$

zaradi $\frac{1}{x^2}$

$\frac{1}{(x+a)^n}, \frac{1}{(x^2+ax+b)^n} \Longrightarrow$ racionalna funkcija, katere imenovalec ima v vsakem faktorju za 1 nižjo stopnjo kot na začetku

$\underline{st(p) < st(g)}$

d) $\int \frac{2dx}{x^3(x^2+1)^2} = \frac{\cancel{Ax^3}+Bx^2+\cancel{Cx}+D}{x^2+(x^2+1)} + E\ln|x| + F\ln|x^2+1| + \cancel{G\operatorname{arctg}x} + H$

LIHA $f(x)$ $\Longrightarrow$ rezultat je lahko samo soda $f(x)$, vse lihe dele izločimo!

$\frac{1}{x} \rightarrow \ln|x|$

$\frac{1}{x^2+1} \rightarrow \ln|x^2+1|$, atg $\frac{2x}{\cancel{\sqrt{4}}}$ $\quad D=0-4=-4$

$$\frac{1}{x^3(x^2+1)^2} = \frac{Ax^3+Bx^2+Cx+D}{x^2(x^2+1)^1}$$

neznano: $\cancel{A}, B, \cancel{C}, D, E, F, \cancel{G}$

$\int$ sode $f(x) \Longrightarrow$ liha $f(x)$

$\int$ lihe $f(x) \Longrightarrow$ soda $f(x)$

$\int f(x)\,dx = g(x) + H$ $\qquad f(x) = \frac{2}{x^3(x^2+1)^2}$

$f(x) = g'(x)$

$$g'(x) = \left(\frac{Bx^2+D}{x^2(x^2+1)} + E\ln|x| + F\ln|x^2+1| + H\right)'$$

$$\frac{2Bx(x^2(x^2+1)) - (Bx^2+D)(2x(x^2+1))}{(x^2(x^2+1))^2} + \frac{E}{x} + \frac{F(2x)}{x^2+1}$$

$$= \frac{\cancel{2x}\,(Bx^2(x^2+1) - (Bx^2+D)(2x^2+1))}{x^{\cancel{4}3}(x^2+1)^2} + \frac{E}{x} + \frac{2Fx}{x^2+1}$$

$$\frac{2}{x^3(x^2+1)^2} = \frac{2(Bx^4+Bx^2-2Bx^4-Bx^2-2Dx^2-D) + Ex^2(x^2+1)^2 + 2Fx\,x^3(x^2+1)}{x^3(x^2+1)^2}$$

izenačimo števce:

DN: $D=-1$
$E=-4$
$F=2$
$B=-2$

e) $\int \frac{x^3}{x^8+3}dx = \int \frac{x^3}{(x^4)^2+3}dt = \frac{1}{4}\int \frac{dt}{t^2+3} =$

$t = x^4$
$dt = 4x^3dx$

$= \frac{1}{4}\frac{1}{\sqrt{3}} \text{atg} \frac{t}{\sqrt{3}} + C$

$\frac{\sqrt{3}}{12} \text{atg} \frac{x^4}{\sqrt{3}} + C$

DN $\int \frac{xdx}{1-x^8}$

R: $\frac{1}{8}\ln|1-x^2| + \frac{1}{8}\ln|1+x^2| + \frac{1}{4}\text{atg}\, x^2 + C$

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f) $\int \frac{x^3}{\underset{(x^4)^2}{x^8}+3}dx = \int \frac{\frac{dt}{4}}{t^2+3} = \frac{1}{4}\int \frac{dt}{t^2+\underset{a^2}{3}} = \frac{1}{4\sqrt{3}} \text{arctg} \frac{t}{\sqrt{3}} + C =$

$= \frac{1}{4\sqrt{3}} \text{arctg} \frac{x^4}{\sqrt{3}} + C$

$t = x^4$
$dt = 4x^3dt$

Določeni integral

$\int f(x)dx = F(x) + C$

$\int_a^b f(x)dx = \underbrace{F(b)}_{\text{zg. meja}} - \underbrace{F(a)}_{\text{sp. meja}}$

1. Izračunaj določeni integral:

a) $\int_{x=0}^{x=\frac{\pi}{4}} \sin x(4x)dx = \int_0^{\pi} \sin t \cdot \frac{dt}{4}$

$\downarrow$
$t = 4x,\ dt = 4dx$

sp. meja $\Rightarrow x = 0 \Rightarrow t = 4 \cdot 0 = 0$
zg. meja $\Rightarrow x = \frac{\pi}{4} \Rightarrow t = 4 \cdot \frac{\pi}{4} = \pi$

$\int \sin x dx = -\cos x + C$

$= \underbrace{\left(-\frac{1}{4}\cos\pi + C\right)}_{\text{zg. meja: } t=\pi} - \underbrace{\left(-\frac{1}{4}\cos 0 + C\right)}_{\text{sp. meja: } t=0} = \frac{1}{4} + \cancel{C} + \frac{1}{4} - \cancel{C} = \frac{1}{2}$

Pri dol. int. se konstante odštejejo.

b) $\int_{-2}^{2} x \sin x \, dx = \left(-x\cos x + \sin x\right)\Big|_{-2}^{2} = -2\cos 2 + \sin 2 - \left(2\cos 2 - \sin 2\right) =$

$= \underline{\underline{-4\cos 2 + 2\sin 2}}$

$u = x \Longrightarrow du = dx$
$dv = \sin x dx \Longrightarrow v = -\cos x$

$\int x \sin x = -x\cos x + \int \cos x \, dx = -x\cos x + \sin x + c$

c) $\int_{1}^{\infty} \frac{dx}{(1+x^2)(3+x^2)} =$ $\Longleftarrow$ POSPLOŠENI INTEGRAL

$\int \frac{dx}{(1+x^2)(3+x^2)} =$

$\frac{1}{(1+x^2)(3+x^2)} = \frac{Ax+B}{1+x^2} + \frac{Cx+D}{3+x^2} = \frac{(Ax+B)(3+x^2)+(Cx+D)(1+x^2)}{(1+x^2)(3+x^2)}$

$1 = 3Ax + Ax^3 + 3B + Bx^2 + Cx + Cx^3 + D + Dx^2$

$x^3: 0 = A + C$
$x^2: 0 = B + D$
$x^1: 0 = 3A + C$
$x^0: 1 = 3B + D$
$-$

$0 = -2A \Rightarrow A = 0, C = 0$
$-1 = -2B \Rightarrow B = \frac{1}{2}, D = -\frac{1}{2}$

$= \int \left(\frac{\frac{1}{2}}{1+x^2} - \frac{\frac{1}{2}}{3+x^2} \right) dx = \frac{1}{2} \operatorname{arctg} x - \frac{1}{2\sqrt{3}} \operatorname{arctg} \frac{x}{\sqrt{3}} + c$

$\int_{1}^{\infty} \frac{dx}{(1+x^2)(3+x^2)} = \left[\frac{1}{2} \operatorname{arctg} x - \frac{1}{2\sqrt{3}} \operatorname{arctg} \frac{x}{\sqrt{3}} \right]_{1}^{\infty} =$

$= \underbrace{\lim_{x\to\infty} \left(\frac{1}{2} \operatorname{arctg} x - \frac{1}{2\sqrt{3}} \operatorname{arctg} \frac{x}{\sqrt{3}} \right)}_{\text{zg. meja}} - \underbrace{\left(\frac{1}{2} \operatorname{arctg} 1 - \frac{1}{2\sqrt{3}} \operatorname{arctg} \frac{1}{\sqrt{3}} \right)}_{\text{sp. meja}} =$

$= \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2\sqrt{3}} \cdot \frac{\pi}{2} - \frac{1}{2} \cdot \frac{\pi}{4} + \frac{1}{2\sqrt{3}} \cdot \frac{\pi}{6} =$

$= \underline{\underline{-\frac{\pi}{6\sqrt{3}} + \frac{\pi}{8}}}$

d) Izračunaj integral $\int_{-2}^{2} f(x)dx$; kjer je $f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 1-|x|; & |x| > 1 \end{cases}$

$$f(x) = \begin{cases} 1-x^2 \; ; & x \in [-1,1] \\ 1-|x| \; ; & x \in (-\infty,-1) \cup (1,\infty) \end{cases}$$

$x \in [-1,1]$

$x \in (-\infty,-1) \cup (1,\infty)$

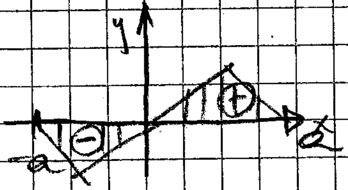
$$\Downarrow \begin{cases} x \; ; & x \ge 0 \\ -x \; ; & x < 0 \end{cases}$$

$$= \begin{cases} 1-x^2 \; ; & x \in [-1,1] \\ 1-x \; ; & x \in (1,\infty) \\ 1+x \; ; & x \in (-\infty,-1) \end{cases}$$

$$\int_{-2}^{2} f(x)dx = \int_{-2}^{-1}(1+x)dx + \int_{-1}^{1}(1-x^2)dx + \int_{1}^{2}(1-x)dx = \underline{\underline{\frac{1}{3}}}$$

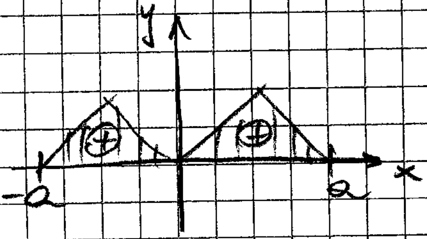
$$\int_{-a}^{a} f(x)dx = 0$$

če je $f(x)$ liha $\Rightarrow \underline{S=0}$

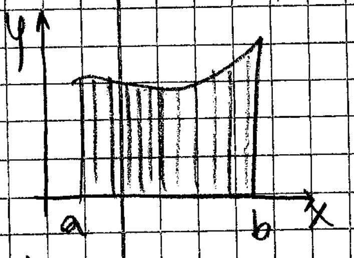


$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

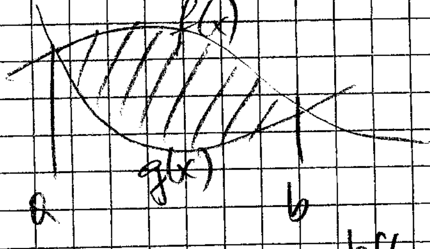
soda



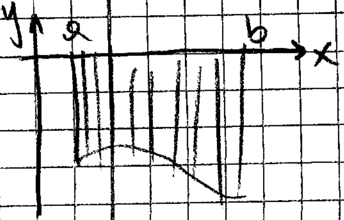
PLOŠČINE



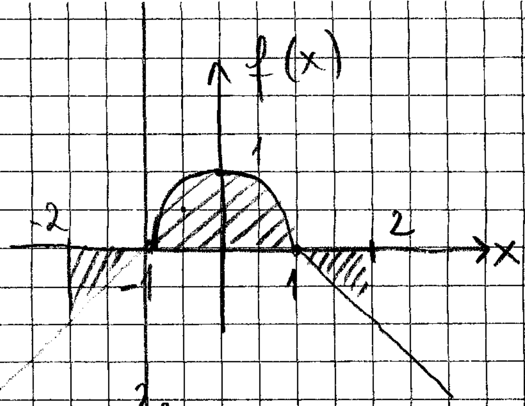
$$S = \int_{a}^{b} f(x)dx$$



$$S = \int_{a}^{b}\left(f(x) - g(x)\right)dx$$



$$S = -\int_{a}^{b} f(x)dx = \int_{b}^{a} f(x)dx$$



$$f(x)=\begin{cases}1-x^2; & x\in[-1,1]\\ 1-x; & x\in(1,\infty)\\ 1+x; & x\in(-\infty,-1)\end{cases}$$

$\int_{-2}^{2} f(x)dx \neq$ ploščina lika

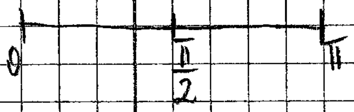
soda

$$S=\frac{1}{2}\cdot 2+\int_{-1}^{1}((1-x^2)-0)\,dx = 1+\int_{-1}^{1}(1-x^2)\,dx =$$

$$1+2\cdot\int_{0}^{1}(1-x^2)\,dx = 1+2\left[x-\frac{x^3}{3}\right]_0^1 = 1+2\left(1-\frac{1}{3}\right) = \frac{7}{3}$$

③ Izračunaj ploščino lika med grafom funkcije $f(x)=\sin x+\cos x+x$, abcisno osjo ter premicama $x=0$, $x=\pi$

predznak:



$\sin x \geq 0$

$\cos x \geq 0$ $\cos x \leq 0$

$x\in[0,\frac{\pi}{2}]$ $x\in[\pi/2,\pi]$

$\uparrow$ + $\uparrow$ +

$$S=\int_0^\pi f(x)dx=\int_0^\pi(\sin x+\cos x+x)\,dx=$$

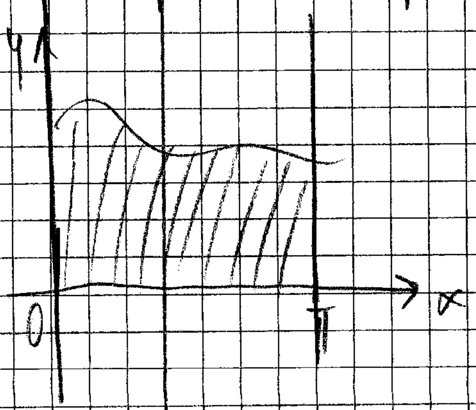
$$=\left[-\cos x+\sin x+\frac{x^2}{2}\right]_0^\pi=$$

$$=\left(-\cos\pi+\sin\pi+\frac{\pi^2}{2}\right)-(-\cos 0+\sin 0+0)=1+\frac{\pi^2}{2}+1=$$

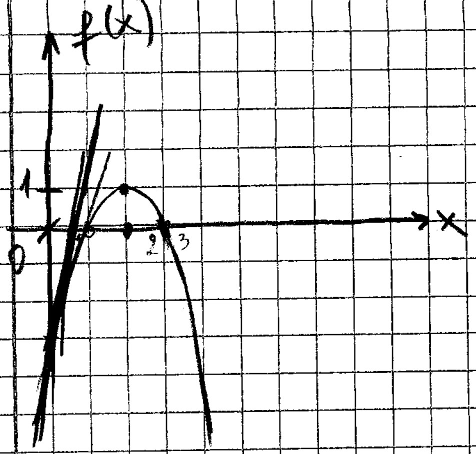
$x\in\emptyset$

$x=\pi$

$$=2+\frac{\pi^2}{2}$$



④ Kolikšna je ploščina lika, ki ga oklepajo parabola $y = -x^2 + 4x - 3$, njena tangenta v točki $(0,-3)$ in x-os



$$y = -x^3 + 4x - 3 = -(x^2 - 4x + 3) =$$

$$= -(x-3)(x-1)$$

$$\left.\begin{array}{l} x_1 = 3 \\ x_2 = 1 \end{array}\right\} \text{ničli}$$

$$T(2,1)$$

tangenta: $k_t = y'(0) = 4$

$$y' = -2x + 4$$

$$\underline{\underline{y = 4x - 3}}$$

$$A(0,-3)$$

$$y = kx + n$$

$$-3 = 4 \cdot 0 + n$$

$$n = -3$$

$$S = S_1 - S_2 = \int_0^1 \left(0 - (-x^2 + 4x - 3)\right) dx - \frac{3}{4} \cdot 3 \cdot \frac{1}{2} =$$

$$= \int_0^1 (x^2 - 4x + 3)\,dx - \frac{9}{8} =$$

$$= \left[\frac{x^3}{3} - 4\frac{x^2}{2} + 3x\right]_0^1 - \frac{9}{8} = \frac{1}{3} - 2 + 3 - \frac{9}{8} = \underline{\underline{\frac{5}{24}}}$$

⑤ Izračunaj S območja $\underbrace{r = \sin^{3/2}\varphi,\ 0 < \varphi < \pi}_{\text{polarne koordinate}}$

$$S = \frac{1}{2}\int_\alpha^\beta r^2 \,\underline{\underline{d\varphi}}$$

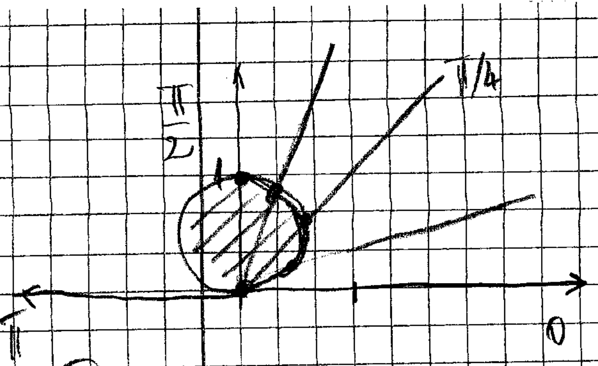
$$S = \frac{1}{2}\int_0^\pi (\sin^{3/2}\varphi)^2 d\varphi = \frac{1}{2}\int_0^\pi \sin^3\varphi \, d\varphi = \frac{1}{2}\left[\frac{1}{3}\cos^3\varphi - \cos\varphi\right]_0^\pi$$

$$\int \sin^3\varphi \, d\varphi = \int \sin^2\varphi \cdot \sin\varphi \, d\varphi = \int (1 - \cos^2\varphi) \sin\varphi \, d\varphi = \int (1 - t^2)(-dt) =$$

$$= \int (t^2 - 1)\,dt = \frac{t^3}{3} - t + C = \frac{\cos^3\varphi}{3} - \cos\varphi + C \qquad \begin{array}{l} \cos\varphi = t \\ dt = -\sin\varphi \, d\varphi \end{array}$$

$$\frac{1}{2}\left[\left(\frac{1}{3}\cos^3\pi - \cos\pi\right) - \left(\frac{1}{3}\cos^3 0 - \cos 0\right)\right] = \frac{1}{2}\left[-\frac{1}{3} + 1 - \frac{1}{3} + 1\right]$$

$$\frac{1}{3} + \frac{1}{3} = \underline{\underline{\frac{2}{3}}}$$



$r = \sin^{3/2} \varphi$

$\varphi = 0 \Rightarrow r = 0$

$\varphi = \frac{\pi}{2} \Rightarrow r = 1$

⑥ Izračunaj ploščino zanke

$x = 2\cos t$
$y = \sin t$

(npr.: cos)

$[0, 2\pi] \begin{cases} x = 2\cos \boxed{t} \\ y = \sin t \end{cases}$

↓ parametrične koordinate

zanka; perioda

$$S = \frac{1}{2}\int_a^b (x\dot{y} - y\dot{x})\,dx$$

$$S = \frac{1}{2}\int_0^{2\pi} (2\cos t \cos t - \sin t\,(-2\sin t))\,dt$$

$$S = \frac{1}{2}\int_0^{2\pi} (2\cos^2 t + \sin^2 t)\,dt =$$

$$= \cancel{\frac{2}{2}}\ 1 \int_0^{2\pi} dt = t\Big|_0^{2\pi} = \underline{\underline{2\pi}}$$

$x = 2\cos t \Rightarrow \cos t = \frac{x}{2}$
$y = \sin t$

$\sin^2 t + \cos^2 t = 1$

$\boxed{y^2 + \frac{x^2}{4} = 1} \Rightarrow$ elipsa

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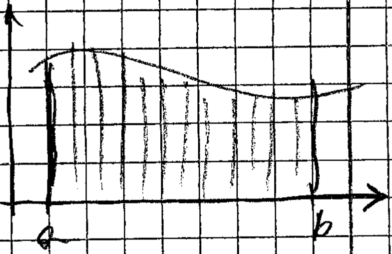
Določeni integral

$\int f(x)dx = F(x) + C$

$\int_a^b f(x)dx = F(x)\Big|_a^b = F(\overset{\text{zg. meja}}{b}) - F(\overset{\text{sp. meja}}{a})$



1. a) $\int_0^{\frac{\pi}{4}} \sin(4x)dx = -\frac{1}{4}\cos(4x)\Big|_0^{\frac{\pi}{4}} = -\frac{1}{4}(-1) - (-\frac{1}{4})(1) = \frac{1}{4} + \frac{1}{4} = \underline{\underline{\frac{1}{2}}}$

$\int \sin(4x)dx = -\frac{1}{4}\cos(4x) + C$

$\int_0^{\frac{\pi}{4}} \sin(4x)dx \Rightarrow \frac{1}{4}\int_0^{\pi} \sin t\,dt = -\frac{1}{4}\cos t\Big|_0^{\pi} = -\frac{1}{4}(\cos\pi - \cos 0) = \frac{1}{2}$

$4x = t$

$4dx = dt$

$dx = \frac{dt}{4}$

sp. meje: $x = 0 \Rightarrow t = 0$

zg. meja: $x = \pi/4 \Rightarrow t = \pi$

b) $\int_{-2}^{2} x\sin x\,dx \Rightarrow \boxed{(\underset{(-1)}{\text{liha } f(x)} \cdot \underset{(-1)}{\text{liha } f(x)} = \underset{①}{\text{soda } f(x)})} \Rightarrow 2\int_0^2 x\sin x\,dx$

$\int x\sin x\,dx = -x\cos x + \int \cos x\,dx =$

$u = x$, $du = dx$

$dv = \sin x\,dx$, $v = -\cos x$

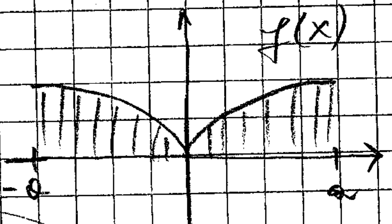
$= -x\cos x + \sin x + C$

soda funkcija

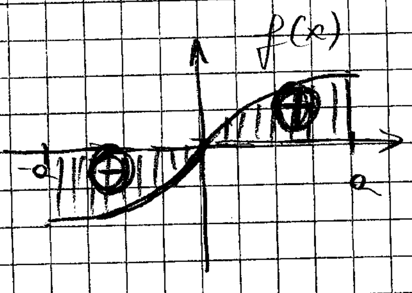
$(-x\cos x + \sin x)\Big|_{-2}^{2} = -2\cos 2 + \sin 2 - (2\cos(-2) + \sin(-2)) =$

$= -2\cos 2 + \sin 2 - 2\cos(2) + \sin 2 = \underline{-4\cos 2 + 2\sin 2}$

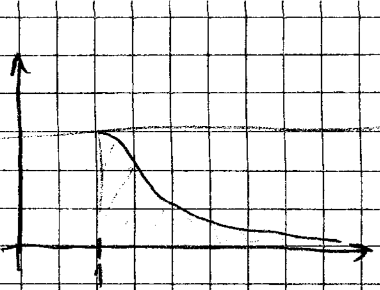
$\int_{-a}^{a} \underset{\text{soda}}{f(x)}\,dx = 2\int_0^a f(x)\,dx$



$\int_{-a}^{a} \underset{\text{liha}}{f(x)}\,dx = 0$



c) $\int_1^{\infty} \frac{dx}{(1+x^2)(3+x^2)}$ } posplošen integral



$\int \frac{dx}{(1+x^2)(3+x^2)} \Rightarrow \frac{1}{(1+x^2)(3+x^2)} = \frac{Ax+B}{(1+x^2)} + \frac{Cx+D}{(3+x^2)}$

$= \frac{1}{(1+x^2)(3+x^2)} = \frac{(Ax+B)(3+x^2)+(Cx+D)(1+x^2)}{(1+x^2)(3+x^2)}$

izenačimo števce:

$x^3: \; 0 = A+C$
$x^2: \; 0 = B+D$
$x^1: \; 0 = 3A+C$
$x^0: \; 1 = D+3B$

$0 = -2A \Rightarrow \boxed{A=0 \quad C=0}$

$-1 = -2B$

$\boxed{B = \frac{1}{2} \quad D = -\frac{1}{2}}$

$a_1 = \sqrt{3}$
$\boxed{a_2 = -\sqrt{3}}$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + c$

$\int \left(\frac{\frac{1}{2}}{(1+x^2)} - \frac{\frac{1}{2}}{(3+x^2)} \right) dx = \frac{1}{2} \arctan x - \frac{1}{2} \frac{1}{\sqrt{3}} \arctan \frac{x}{\sqrt{3}} + C =$

$= \left(\frac{1}{2} \arctan x - \underbrace{\frac{\sqrt{3}}{6} \arctan \frac{x}{\sqrt{3}}}_{\text{liha funkcija}} \right) \Big|_1^{\infty} =$

$\left(-\frac{\sqrt{3}}{6} \arctan \frac{x}{(-\sqrt{3})} \right)$
$\left(\frac{\sqrt{3}}{6} \arctan \frac{x}{\sqrt{3}} \right)$

$= \underbrace{\lim_{x \to \infty} \left(\frac{1}{2} \arctan x - \frac{\sqrt{3}}{6} \operatorname{arctg} \frac{x}{\sqrt{3}} \right)}_{\text{zg. meja}} - \underbrace{\left(\frac{1}{2} \operatorname{arctg} 1 - \frac{\sqrt{3}}{6} \operatorname{arctg} \frac{\sqrt{3}}{3} \right)}_{\text{sp. meja}} =$

$= \left(\frac{1}{2} \frac{\pi}{2} - \frac{\sqrt{3}}{6} \cdot \frac{\pi}{2} \right) - \left(\frac{1}{2} \cdot \frac{\pi}{4} - \frac{\sqrt{3}}{6} \cdot \frac{\pi}{6} \right) = \left(\frac{\pi}{4} - \frac{\sqrt{3}}{12} \pi \right) - \left(\frac{\pi}{8} - \frac{\sqrt{3} \cdot \pi}{36} \right) =$

$\frac{2\pi}{8} - \frac{\pi}{8} - \frac{3\sqrt{3}}{36} \pi + \frac{\sqrt{3}\,\pi}{36} = \frac{\pi}{8} - \frac{2\sqrt{3}}{36} \pi = \underline{\underline{\frac{\pi}{8} - \frac{\sqrt{3}}{18} \pi}}$

d) $f(x) = \begin{cases} 1-x^2 \; ; & |x| \leq 1 \\ 1-|x| \; ; & |x| > 1 \end{cases} \qquad \int_{-2}^{2} f(x)\, dx$

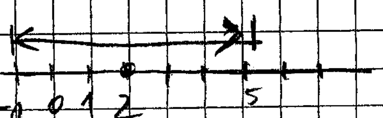
$|x| = \begin{cases} x \; ; & x \geq 0 \\ -x \; ; & x < 0 \end{cases}$

$|x| \leq 1 \; ; \; x \in [-1, 1]$

$x \in (-\infty, -1) \cup (1, \underset{+}{\infty})$

$$f(x)=\begin{cases}1-x^2, & x\in[-1,1]\\ 1-x, & x\in(1,\infty)\\ 1+x, & x\in(-\infty,-1)\end{cases}$$

$|x-2|<3$

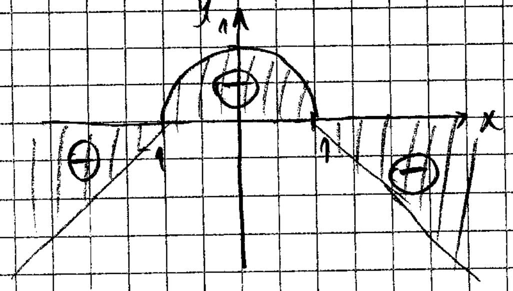


$x\in[-1,5]$

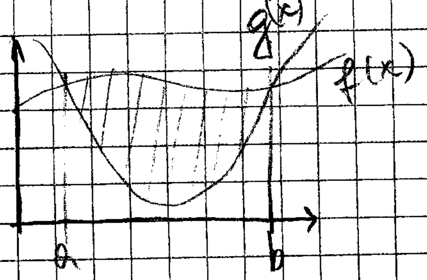
$$\int_{-2}^{2}f(x)\,dx=\int_{-2}^{-1}(1+x)\,dx+\int_{-1}^{1}(1-x^2)\,dx+\int_{1}^{2}(1-x)\,dx=$$

$$=\left(x+\frac{x^2}{2}\right)\Big|_{-2}^{-1}+\left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1}+\left(x-\frac{x^2}{2}\right)\Big|_{1}^{2}=\left(\frac{3}{2}-\cancel{(2-2)}\right)+\left(+1-\frac{1}{3}-(-\cancel{1}+\frac{1}{3}\right)+\cancel{2-2}-(1$$

$$=\frac{3}{2}+\left(2-\frac{2}{3}\right)-\frac{1}{2}=-\frac{3}{2}+\frac{4}{3}+\frac{1}{2}=\frac{4}{3}-1=\frac{1}{3}\ ?$$



PLOŠČINE



$$S=\int_{a}^{b}(\underbrace{f(x)}_{\text{zg. krivulja}}-\underbrace{g(x)}_{\text{sp. krivulja}})\,dx$$

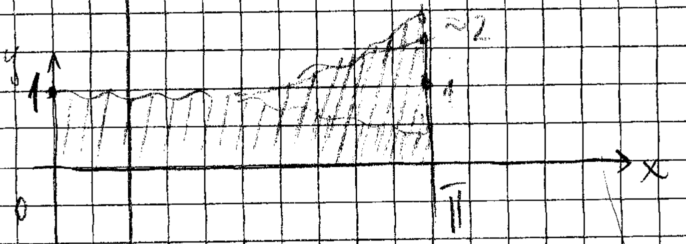
1. Izračunaj ploščino območja med funkcijo $f(x)=\begin{cases}1-x^2; & |x|\le 1\\ 1-|x|; & |x|>1\end{cases}$, abscisno osjo in premicamo $x=-2$ in $x=2$

$$S=\underbrace{\int_{-2}^{-1}(0-(1+x))\,dx}_{\text{pravokotni}}+\int_{-1}^{1}(1-x^2-0)\,dx+\underbrace{\int_{1}^{2}(0-(1-x))\,dx}_{\text{pravokotni}}$$

$\triangle$ $\parallel$ $\frac{1}{2}$ $\qquad = 2\int_{0}^{1}(1-x^2)\,dx \qquad$ $\triangle$ $\frac{1}{2}$

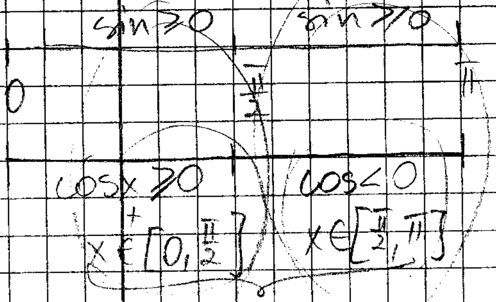
$$1+\left(2x-\frac{2x^3}{3}\right)\Big|_{0}^{1}=1+\left(2-\frac{2}{3}-0\right)=1+\left(\frac{4}{3}\right)=\frac{7}{3}$$

2. Izračunaj ploščino lika med grafom funkcije $f(x) = \sin x + \cos x + x$, abscisno osjo in premicama $x=0$ ter $x=\pi$



predznak:

$f(x) = \sin x + \cos x + x$



$f(x) \geq 0$

$$S = \int_0^{\pi} (f(x) - 0)\,dx = \int_0^{\pi} (\sin x + \cos x + x)\,dx =$$

$$= -\cos x + \sin x + \frac{x^2}{2}\Big|_0^{\pi} = \left(-\cos\pi + \sin\pi + \frac{\pi^2}{2}\right) -$$

$$-(-\cos 0 + \sin 0 + 0) = 1 + \frac{\pi^2}{2} + 1 = \boxed{2 + \frac{\pi^2}{2}}$$

3. Kolikšna je ploščina lika, ki ga oklepajo parabola $y = -x^2 + 4x - 3$, njena tangenta v točki $(0, -3)$ in x-os?

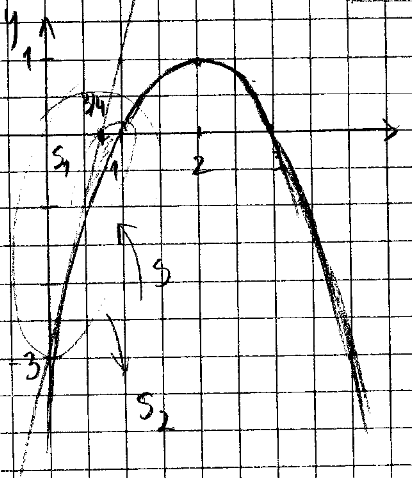
$-x^2 + 4x - 3 = -4 + 8 - 3 = 1$

$-2x + 4$

$$\boxed{\begin{array}{l} x = 2 \\ y = 1 \end{array}}$$

n: $-(x-3)(x-1) = 0$

$$\boxed{\begin{array}{l} x_1 = 3 \\ x_2 = 1 \end{array}}$$



$k = f'(0) = 4$

$A(0, -3)$

$-3 = 4 \cdot 0 + n$

$n = -3$

$y = 4x - 3$

$4x - 3 = 0$

$$\boxed{x = \frac{3}{4}}$$

I. $S = S_2 - S_1 = \int_0^1 (0 - (-x^2 + 4x - 3))\,dx - \frac{3}{4} \cdot 3 \cdot \frac{1}{2} =$

$$= \frac{x^3}{3} - \frac{4x^2}{2} + 3x\Big|_0^1 - \frac{9}{8} = \frac{1}{3} - 2 + 3 - \frac{9}{8} = \frac{4}{3} - \frac{9}{8} = \frac{32 - 27}{24} = \boxed{\frac{5}{24}}$$

II. $S = \int_0^{3/4} ((4x-3) - (\text{parabola}))\,dx + \int_{3/4}^{1} (0 - (\text{parabola}))\,dx$

$$= 2\int_0^1 \mathrm{ch}x\,dx = 2\mathrm{sh}x\Big|_0^1 = 2\mathrm{sh}1 - 2\overbrace{\mathrm{sh}0}^{0} = \cancel{2}\,\frac{e^1-e^{-1}}{\cancel{2}} = e-\frac{1}{e}$$

2. DN Izračunaj dolžino loka krivulje $y^2=x^3$ med presečiščema s premico $2y=x$

R: $\frac{61}{216}$

3. Izračunaj dolžino zanke $r=a(1+\cos\varphi)$, $a>0$

R: $8a$ $\varphi \in [0,2\pi]$

4. Izračunaj dolžino loka krivulje

$x=\frac{t^6}{6}$, $y=2-\frac{t^4}{4}$, $y>0$, $t>0$

$2-\frac{t^4}{4}>0$

$2>\frac{t^4}{4}$

$8>t^4$

$t<\sqrt[4]{8}$

$$s=\int_0^{\sqrt[4]{8}}\sqrt{(\dot{x}^2+\dot{y}^2)}\,dt=\int_0^{\sqrt[4]{8}}\sqrt{(\tfrac{1}{\cancel{6}}\cancel{6}t^5)^2+(-\tfrac{1}{\cancel{4}}\cancel{4}t^3)^2}\,dt=$$

$$s=\int_0^{\sqrt[4]{8}}\sqrt{t^{10}+t^6}\,dt=\int_0^{\sqrt[4]{8}}\sqrt{t^6(t^4+1)}\,dt=$$

$$=\int |t^3|\sqrt{t^4+1}\,dt=\int t^3\sqrt{t^4+1}\,dt=$$

$\boxed{\text{meje so} > 0}$

$t^4+1=u$

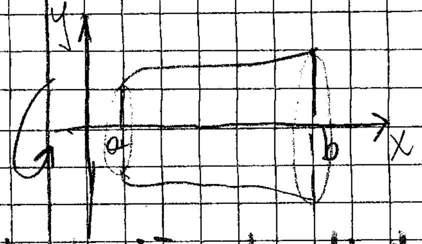
$du=4t^3\,dt$

$$=\int \sqrt{u}\,\frac{du}{4}=\frac{1}{4}\int\sqrt{u}\,du=\frac{1}{4\cdot 2}\cdot\frac{2}{3}u^{3/2}=$$

$$\frac{1}{6}\sqrt{u^3}=\frac{1}{6}\sqrt{(t^4+1)^3}+C \Rightarrow$$

$$\Rightarrow \frac{1}{6}\sqrt{(t^4+1)^3}\Big|_0^{\sqrt[4]{8}}=\frac{1}{6}\left(\sqrt{9^3}-1\right)=\frac{26}{6}=\frac{13}{3}$$

Vrtenina



kartezične koordinate:

$$V = \pi\int_a^b y^2\,dx \qquad P = 2\pi\int_a^b y\cdot\sqrt{1+(y')^2}\,dx$$

polarne koordinate:

$$V = \pi\int_\alpha^\beta r^2\,d\varphi \qquad P = 2\pi\int_\alpha^\beta r\sin\varphi\sqrt{(r')^2+r^2}\,d\varphi$$

parametrične koordinate:

$$V = \pi\int_a^b y^2\dot{x}\,dt \qquad P = 2\pi\int_a^b y\underbrace{\sqrt{\dot{x}^2+\dot{y}^2}}_{\text{dolžina loka}}\,dt$$

1. Izračunaj volumen telesa, ki nastane, če krivuljo zavrtimo okrog x-osi

$y = 3\sqrt{(1-x)^3 x}$

$0 < x < 1$

$$V = \pi\int_0^1 y^2\,dx = \pi\int_0^1 9(1-x)^3 x\,dx =$$

$$= 9\pi\int^1 (x - 3x^2 + 3x^3 - x^4)\,dx =$$

$$= 9\pi\left(\frac{x^2}{2} - x^3 + \frac{3}{4}x^4 - \frac{x^5}{5}\right)\Big|_0^1 = 9\pi\left(1/2 - 1 + \frac{3}{4} - \frac{1}{5}\right) = 9\pi\left(\frac{1}{4} - \frac{1}{5}\right) =$$

$$= 9\pi\left(\frac{5-4}{20}\right) = \underline{\underline{\frac{9\pi}{20}}}$$

2. Izračunaj površino vrtenine, ki nastane, če funkcijo zavrtimo okoli x osi

$f(x) = \sqrt{1-x^2}$

$0 < x < 1$

$$2\pi\int_0^1 \sqrt{1-x^2}\cdot\sqrt{1+\left(\frac{x}{\sqrt{1-x^2}}\right)^2}\,dx =$$

$$f(x)\sqrt{1+y^2} = \sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}} = \cancel{\sqrt{1-x^2}}\cdot\frac{\sqrt{1-x^2+x^2}}{\cancel{\sqrt{1-x^2}}} = 1$$

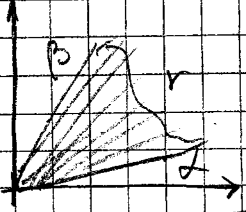
$$2\pi\int_0^1 1\,dx = 2\pi x\Big|_0^1 = \underline{\underline{2\pi}}$$

4. Izračunaj ploščino območja

$r=\sin^{3/2}\varphi\ ,\ 0<\varphi<\pi$

polarne koordinate

$$\boxed{S=\frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\varphi}$$



$$S=\frac{1}{2}\int_0^{\pi}\sin^3\varphi\,d\varphi$$

$\cos\varphi=t$

$dt=-\sin\varphi\,d\varphi$

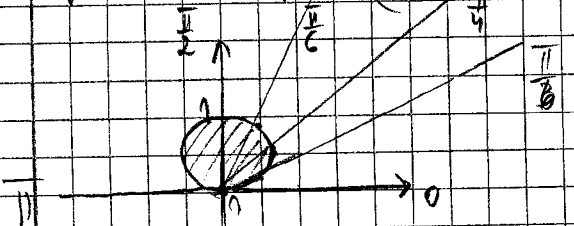
$$\int\sin^3\varphi\,d\varphi=\int\sin^2\varphi\,\sin\varphi\,d\varphi=\int(1-\cos^2\varphi)\sin\varphi\,d\varphi=$$

$$=-\int(1-t^2)\,dt=-t+\frac{t^3}{3}=\frac{t^3}{3}-t=\frac{1}{3}\cos^3\varphi-\cos\varphi+C$$

$$=\frac{1}{2}\left(\frac{1}{3}\cos^3\varphi-\cos\varphi\right)\Big|_0^{\pi}=\frac{1}{2}\left(\left(\frac{1}{3}(-1)-(-1)\right)-\left(\frac{1}{3}\cdot 1-1\right)\right)=$$

$$=\frac{1}{2}\left(-\frac{1}{3}+1-\frac{1}{3}+1\right)=\frac{1}{2}\left(\frac{2}{3}+2\right)=\frac{2}{3}$$

$r=\sin^{3/2}\varphi=(\sin\varphi)^{3/2}$



$\varphi=0\ ;\ r=0$

$\varphi=\frac{\pi}{2}\ ,\ r=1$

$\varphi=\frac{\pi}{6}\ ,\ \sqrt{\frac{1}{8}}=\frac{1}{2\sqrt{2}}=\frac{\sqrt{2}}{4}$

$\varphi=\frac{\pi}{4}\ ,\ \left(\frac{\sqrt{2}}{2}\right)^{3/2}$

5. Izračunaj ploščino zanke

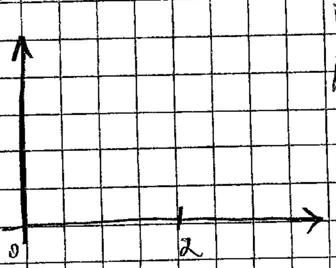
$x=2\cos t$

$y=\cos t$

$\left.\begin{array}{l}x=2\cos t\\ y=\sin t\end{array}\right\}$ parametrične koordinate

$$\boxed{S=\frac{1}{2}\int_a^b(x\dot{y}-y\dot{x})\,dt}$$

$t=0\ ,\ x=2\ ,\ y=0$



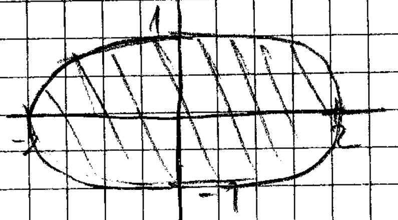
$t=[0,2\pi]$

perioda $\sin x$ in $\cos x$ $=2\pi$

$$S=\frac{1}{2}\int_0^{2\pi}\left(2\cos t\cdot\cos t-\sin t\,(-2\sin t)\right)dt=$$

$$=\frac{1}{2}\int_0^{2\pi}\underbrace{(2\cos^2t+2\sin^2t)}_{1}dt=\frac{1}{2}\int_0^{2\pi}2\,dt=\frac{1}{\cancel{2}}\cancel{2}t\Big|_0^{2\pi}=2\pi$$

$$\sin^2t+\cos^2t=1$$
$$y^2+\left(\frac{x}{2}\right)^2=1$$
$$y^2+\frac{x^2}{4}=1$$



Dolžina loka krivulje

- kartezične koordinate:

$$s=\int_a^b\sqrt{1+(y')^2}\,dx$$

- polarne koordinate:

$$s=\int_\alpha^\beta\sqrt{(r')^2+r^2}\,d\varphi$$

- parametrične koordinate:

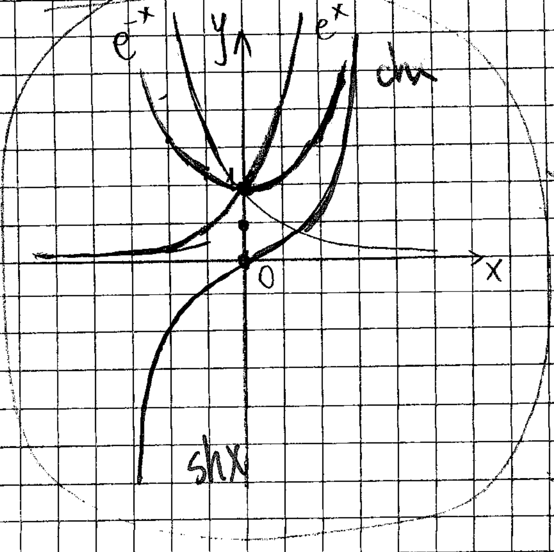
$$s=\int_a^b\sqrt{\dot{x}^2+\dot{y}^2}\,dt$$

$$chx=\frac{e^x+e^{-x}}{2}$$ hiperbolični kosinus

$$shx=\frac{e^x-e^{-x}}{2}$$ hiperbolični sinus

$$(shx)'=chx$$
$$(chx)'=shx$$
$$\int shx\,dx=chx+c$$
$$\int chx\,dx=shx+c$$
$$1+sh^2x=ch^2x$$

1. Izračunaj dolžino loka krivulje $y=chx$ $-1<x<1$



$$s=\int_{-1}^{1}\sqrt{1+(y')^2}\,dx=\int_{-1}^{1}\sqrt{1+sh^2x}\,dx=$$

soda f(x)

$$=\int_{-1}^{1}\sqrt{ch^2x}\,dx=\int_{-1}^{1}|chx|\,dx=\int_{-1}^{1}chx\,dx=$$

11. 1. 2011

nastavek

$$\int \frac{p(x)}{(ax^2+bx+c)(x+d)}\,dx = \underbrace{A\ln|x+d|}_{\text{linearni faktor}} \underbrace{+B\ln|ax^2+bx+c| + C\operatorname{arctg}\frac{2ax+b}{\sqrt{-D}}}_{\text{kvadratni faktor}} + K$$

$D = b^2-4ac$

$\underline{D<0}$

$st(p(x)) \leq 2$

$$\int \frac{3x^2+5x+6}{(x+2)(x^2+x+2)} = A\ln|x+2| + B\ln|x^2+x+2| + C\operatorname{arctg}\frac{2x+1}{\sqrt{7}} + K$$

$D = 1-8 = -7$

odvajamo:

$$\frac{3x^2+5x+6}{(x+2)(x^2+x+2)} = \frac{A}{x+2} + \frac{B(2x+1)}{x^2+x+2} + \frac{C\,\frac{2}{\sqrt{7}}}{1+\frac{(2x+1)^2}{7}}$$

$$\frac{C\,\cancel{14}\,2\cdot\sqrt{7}}{(7+(2x+1)^2)\cancel{\sqrt{7}}} = \frac{2C\sqrt{7}}{4x^2+4x+8} = \frac{\cancel{2}C\sqrt{7}}{\cancel{4}\cdot 2(x^2+x+2)} =$$

$$= \frac{A(x^2+x+2) + B\cdot(2x+1)(x+2) + \frac{C\sqrt{7}}{2}(x+2)}{(x+2)(x^2+x+2)} =$$

$$= \frac{Ax^2+Ax+2A+2Bx^2+5Bx+2B+\frac{\sqrt{7}}{2}Cx+C\sqrt{7}}{(x+2)(x^2+x+2)}$$

x^2: $3 = A+2B$

x^1: $5 = A+5B+\frac{\sqrt{7}}{2}C \quad /(-2)$

x^0: $6 = 2A+2B+C\sqrt{7}$

$-10 = -2A-10B-\sqrt{7}C$

$\underline{6 = 2A+2B+C\sqrt{7}}$

$-4 = -8B$

$\boxed{B=\frac{1}{2}}$

$6 = 4+1+C\sqrt{7}$

$\boxed{C=\frac{1}{\sqrt{7}}}$

$A+1=3$

$\boxed{A=2}$

$$2\ln|x+2| + \frac{1}{2}\ln|x^2+x+2| + \frac{1}{\sqrt{7}}\operatorname{arctg}\frac{2x+1}{\sqrt{7}}$$

$f(x) = x^2e^{2x} + \frac{2x}{x^2+4}$

$\int x^2e^{2x}dx + \int \frac{2x}{x^2+4}dx$

$u = x^2 \qquad du = 2x\,dx$

$dv = e^{2x}dx \qquad v = \frac{1}{2}e^{2x}$

$u = x \qquad du = dx$

$dv = e^{2x}dx \qquad v = \frac{1}{2}e^{2x}$

$\frac{1}{2}x^2e^{2x} - \cancel{\frac{1}{2}}\cancel{2}\int e^{2x}x\,dx =$

$= \frac{1}{2}x^2e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx\right)$

$\frac{1}{2}x^2e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right)$

$\frac{1}{2}e^{2x}\left(x^2 - x + \frac{1}{2}\right) + \ln|x^2+4| + C$

$y = \sqrt{2x^3}$

$0 \leq x \leq \frac{2}{3}$

dolžina loka

$s = \int_a^b \sqrt{1+(y')^2}\,dx$ $\quad y' = \boxed{\sqrt{2} \cdot \frac{3}{2} x^{1/2}}$

$s = \int_0^{\frac{2}{3}} \sqrt{1 + \cancel{2} \cdot \frac{9}{\cancel{4}_2} x}\,dx$

$s = \int_0^{\frac{2}{3}} \sqrt{1 + \frac{9}{2} x}\,dx$

$t = 1 + \frac{9}{2} x$

$dt = \frac{9}{2} dx$

$dx = \frac{2}{9} dt$

$x = 0 \Rightarrow 1$

$x = \frac{2}{3} \Rightarrow 4$

$\frac{2}{9} \int_1^4 \sqrt{t}\,dt = \frac{2}{9} \cdot \frac{2}{3} \cdot t^{3/2}$

$\frac{4}{27} t^{3/2} \Big|_1^4 = \frac{4}{27}(8-1) = \frac{28}{27}$

$f(x) = \sqrt{\frac{1}{x^2-x}}$

$x \in [2,3]$

$V = \pi \int_a^b f^2(x)\,dx = \pi \int_2^3 \left(\sqrt{\frac{1}{x^2-x}}\right)^2 dx = \pi \int_2^3 \frac{1}{x(x-1)}\,dx$

$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{Ax - A + Bx}{x(x-1)}$

$x^1: \; 0 = A + B$

$x^0: \; 1 = -A$

$A = -1$

$B = 1$

$\int \left(\frac{-1}{x} + \frac{1}{x-1}\right) dx$

$-\ln|x| + \ln|x-1| \Big|_2^3 = -\ln|3| + \ln|2| - (-\ln|2| + 0$

$= -\ln|3| + 2\ln|2| = \underline{2\ln 2 - \ln 3}$

$\pi (2\ln 2 - \ln 3)$

$\pi \ln \frac{4}{3}$

$y = 0$

$f(x) = x^2 + 2x - 3$

$T(2, f(2))$

$T(2, 5)$

$f'(x) = 2x + 2$

$f'(2) = 6 = k_t$

$k_n = -\frac{1}{6}$

$y = kx + n$

$5 = -\frac{1}{3} + n$

$n = 5 + \frac{1}{3} = \frac{16}{3}$

$y = -\frac{1}{6}x + \frac{16}{3}$

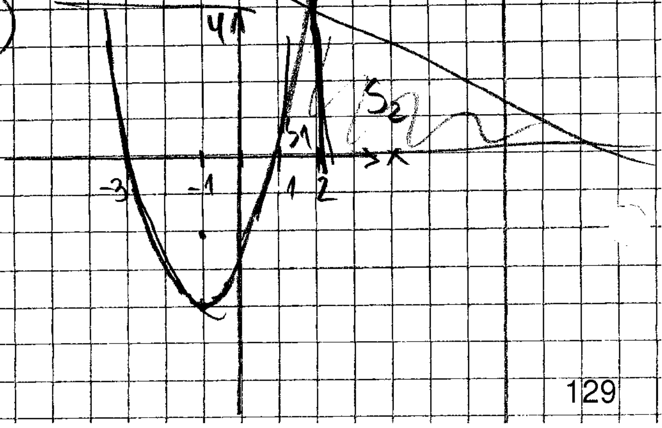
$x^2 + 2x - 3 = -\frac{1}{6}x + \frac{16}{3} \quad / \cdot 6$

$6x^2 + 12x - 18 + x - 32 = 0$

$6x^2 + 13x - 50 = 0$

$y = (x+3)(x-1)$

$T(-1, -4)$



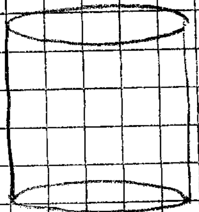
$\int_1^2 (x^2 + 2x - 3)\,dx = \frac{x^3}{3} + x^2 - 3x \Big|_1^2 = \frac{7}{3}$

$S_2 = \frac{1}{2} \cdot 5 \cdot 30 = 5 \cdot 15 = 75$ $\quad S = S_1 + S_2$

$P = 8\pi$

$V = max$

Valjasto posoda s pokrovom



$V = \pi\, O \cdot v = \pi r^2 \cdot v$

$P = 2O + Pl = \cancel{2\pi} r^2 + \cancel{2\pi} r \cdot v = 8\pi$

$v = \frac{4 - r^2}{r}$

$V(r) = \pi r^2 \cdot \frac{4 - r^2}{r} = \pi r (4 - r^2) = \underline{\underline{4\pi r - \pi r^3}}$

$V'(r) = 4\pi - 3\pi r^2$

$4\pi - 3\pi r^2 = 0$

$4\cancel{\pi} = 3\cancel{\pi} r^2$

$r^2 = \frac{4}{3}$

$r = \underline{\underline{\frac{2\sqrt{3}}{3}}}$

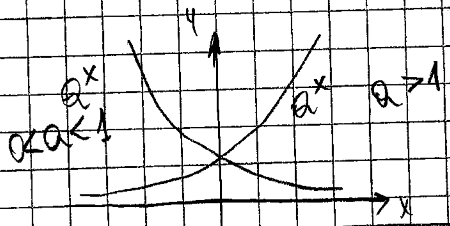
$V''(r) = -6\pi r$

$V''\left(\frac{2\sqrt{3}}{3}\right) < 0 \Leftarrow$ maksimum

$V = \frac{4 - \frac{4}{3}}{\frac{2\sqrt{3}}{3}} = \frac{4\sqrt{3}}{3}$

$\boxed{\text{Liha funkcija na simetričnem intervalu} \Rightarrow \text{Ploščina} = 0}$

$$\lim_{x \to \infty} \frac{2^{x+1} + 7^{x+1}}{2^x + 7^x} = \lim_{x \to \infty} \frac{\left(\frac{2}{7}\right)^{x+1} + 1}{\frac{2^x}{7^{x+1}} + \frac{7^x}{7^{x+1}}} = \lim_{x \to \infty} \frac{\cancel{\left(\frac{2}{7}\right)^{x+1}}^{0} + 1}{\frac{1}{7}\cancel{\left(\frac{2}{7}\right)^{x}}^{0} + \frac{1}{7}} = \underline{\underline{7}}$$



$$\lim_{x \to 0} \frac{\arctan 3x}{7x} = \frac{0}{0}$$

$$\frac{\frac{3}{1+9x^2}}{7} = \frac{3}{7}$$

12. 1. 2010

① Določite limito

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\infty-\infty=?$$

$$\lim_{x\to 0}\left(\frac{\sin x-x}{x\sin x}\right)=\frac{0}{0}\qquad \boxed{\text{L'Hospital}\ \frac{0}{0},\ \pm\frac{\infty}{\infty}}$$

$$\lim_{x\to 0}\left(\frac{\cos x-1}{\sin x+x\cos x}\right)=\frac{1-1}{0+0}=\frac{0}{0}$$

$$\lim_{x\to 0}\left(\frac{-\sin x}{+\cos x+\cos x-x\sin x}\right)=\lim_{x\to 0}\left(\frac{-\sin x}{2\cos x-x\sin x}\right)=\frac{0}{2}=\underline{\underline{0}}$$

② Izračunaj odvod implicitno podane funkcije:

$$y^{\sin x}=\ln(x^2+y^2)\ \Big/\ (\ )'$$

$$\boxed{e^{\ln}}y^{\sin x}=\ln(x^2+y^2)$$

$$e^{\sin x\ln y}=\ln(x^2+y^2)\ \Big/\ (\ )'$$

$$e^{\sin x\ln y}\cdot(\sin x\ln y)'=\frac{1}{x^2+y^2}(x^2+y^2)'$$

$$e^{\sin x\ln y}\cdot\left(\cos x\ln y+\frac{\sin x}{y}\cdot y'\right)=\frac{1}{x^2+y^2}\cdot(2x+2yy')\ \Big/\ (x^2+y^2)y$$

$$y^{\sin x}(y\cdot\cos x\ln y+\sin x\cdot y')(x^2+y^2)=y(2x+2yy')$$

$$y^{\sin x}\sin x\,y'(x^2+y^2)-2y^2y'=2xy-y^{\sin x}\cdot y\cos x\ln y(x^2+y^2)$$

$$y'\left(y^{\sin x}\sin x(x^2+y^2)-2y^2\right)=2xy-y^{\sin x+1}\cdot\cos x\ln y(x^2+y^2)$$

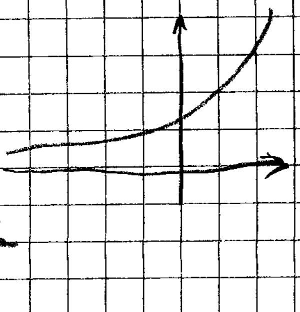
$$y'=\frac{2xy-y^{\sin x+1}\cos x\ln y(x^2+y^2)}{y^{\sin x}\sin x(x^2+y^2)-2y^2}$$

③ Poišči tisto tangento na graf funkcije, ki je vzporedna s premico

$f(x)=e^{2x}-x$

$y+x+1=0\Rightarrow y=-x-1$

$k_t=f'(x)=\underline{e^{2x}\cdot 2-1}$



$k_t=k$

$2e^{2x}-1=-1$

$e^{2x}=\emptyset$

nima rešitve, taka tangenta ne obstaja

④ Podana je funkcija $f(x)$

a) Določi ničle, pole, asimptote in ekstreme funkcije $f(x)$ in nariši graf
b) Določi Dg: $g(x) = \ln(f(x))$ in nariši graf

a) N: $x^2+2x+1=0$
$(x+1)^2=0$
$x=-1$ (2.st)

P: $x^2-4=0$
$(x-2)(x+2)=0$
$x_1=2$
$x_2=-2$

A: $(x^2+2x+1):x^2-4=1$
$\underline{-x^2\ \ -4}$
$2x+5$
$x=-\frac{5}{2}$
seka asimptoto

$\boxed{y=1}$

$$f'(x)=\left(\frac{x^2+2x+1}{x^2-4}\right)'=$$

$$=\frac{(2x+2)(x^2-4)-(x^2+2x+1)(2x)}{(x^2-4)^2}=$$

$$=\frac{2(x+1)(x^2-4-(x+1)\cdot x)}{(x^2-4)^2}=\frac{2(x+1)(-x-4)}{(x^2-4)^2}=0$$

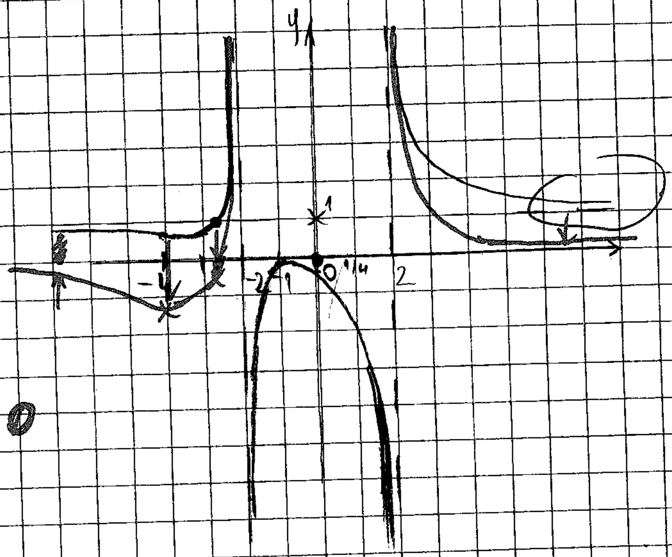
$$-2(x+1)(x+4)=0$$

$x_1=-1$ $y_1=0$
$x_2=-4$ $y_2=\frac{9}{12}=\frac{3}{4}$

$16-8+1=9$

$P_y(0,-\frac{1}{4})$

ln1=0
točka se premakne
$\ln\frac{3}{4}<0$

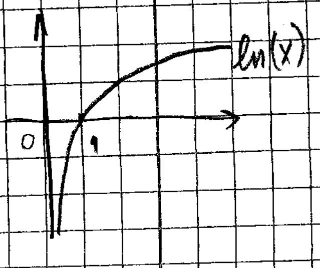


$Z_g=[\ln 3/4,\infty)$

$Z_f\ (-\infty,0]\cup[\frac{3}{4},\infty)$

b) Dg: $g(x)=\ln(f(x))$

$Dg=\{x, f(x)>0\}=\underline{(-\infty,-2)\cup(2,\infty)}=R-[-2,2]$



⑤ Določi Df ~~g~~(x), nariši graf in določi točke, kjer funkcija g(x) doseže največjo oziroma najmanjšo vrednost na intervalu [0, 2].

Kolikšni sta največja in najmanjša vrednost g(x) na [0, 2]?

$$g(x) = x^3(2 - x\sqrt{x})$$

$$D_g = \{x; x \geq 0\} = [0, \infty)$$

N: $x^3(2 - x\sqrt{x}) = 0$

$x_1 = 0$ (3. st.)

$2 - x\sqrt{x} = 0$

$2 - x^{3/2} = 0$

$x^{3/2} = 2$

$x = 2^{2/3} = \sqrt[3]{4}$

E: $3x^2(2 - x^{3/2}) + x^3\left(-\frac{3}{2}x^{1/2}\right)$

$\frac{3}{2}x^2(4 - 2x^{3/2} - x^{3/2}) = \frac{3}{2}x^2(4 - 3x^{3/2})$

$\frac{3}{2}x^2(4 - 3x^{3/2}) = 0$

$x_1 = 0$

$x_2 = \quad 4 - 3x^{3/2} = 0$

$x^{3/2} = \frac{4}{3}$

$$\underline{\underline{x = \left(\frac{4}{3}\right)^{2/3} = \sqrt[3]{\frac{16}{9}}}}$$

intervali naraščanja in padanja

$f'(x) \geq 0$, naraščanje

$\frac{3}{2}x^2(4 - 3x^{3/2}) \geq 0$

vedno pozitivno

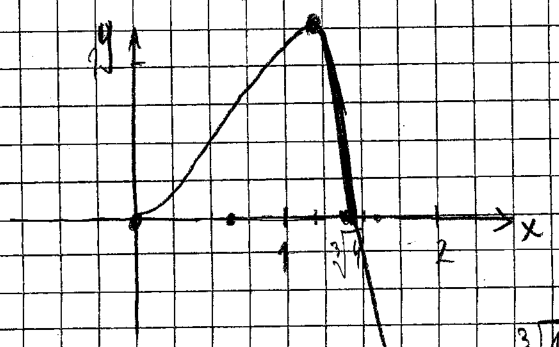
$\Downarrow$

$4 - 3x^{3/2} \geq 0$

$x^{3/2} \leq \frac{4}{3}$

$x \leq \sqrt[3]{\frac{16}{9}}$ narašča

$x > \sqrt[3]{\frac{16}{9}}$ pada



$f\left(\sqrt[3]{\frac{16}{9}}\right) = \frac{16}{9}\left(2 - \frac{4}{3}\right) = \frac{16}{9} \cdot \frac{2}{3} = \frac{32}{27}$

$\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \underset{\underset{\infty}{\downarrow}}{x^3}(2 - \underset{\underset{-\infty}{\downarrow}}{x^{3/2}}) = \infty \cdot (-\infty) = -\infty$

največja vrednost $x = \sqrt[3]{\frac{16}{9}} \Rightarrow y = \frac{32}{27}$

najmanjša vrednost $x = 2 \Rightarrow f(2) = 8(2 - 2^{3/2}) = 2^3(2 - 2^{3/2}) = 2^4 - 2^{9/2} =$

$= 2^4(1 - \sqrt{2}) \doteq (-6 \sim -7)$

⑥ $\int \sin^3x\cos^3x\,dx = \int \sin x\,\sin^2x\cos^3x\,dx = \int \sin x\,(1-\cos^2x)\cos^3x\,dx$

$t=\cos x$
$dt=-\sin x\,dx$

$= + \int (1-t^2)\,t^3\,dx = \int (t^5-t^3)\,dt = \frac{t^6}{6}-\frac{t^4}{4}+C$

$$\underline{\underline{\frac{1}{6}\cos^6x-\frac{1}{4}\cos^4x+C}}$$

---

⑦ Izračunajte nedoločeni integral:

$$\int\left(\frac{x^3-2x+1}{4+x^2}+\frac{3x+2}{x^2-1}\right)dx = \int\frac{x^3-2x+1}{4+x^2}dx+\int\frac{3x+2}{x^2-1}dx$$

$x^3 \quad -2x+1 : x^2+4 = x$
$-x^3 \pm 4x$
$\quad -6x+1$

$$\int\left(x+\frac{-6x+1}{(4+x^2)}\right)dx = \frac{x^2}{2}+\int\frac{-6x+1}{4+x^2}dx$$

$x=\frac{1}{6}$

$$\int\frac{-6x}{4+x^2}dx-\int\frac{1}{4+x^2}dx = \int\frac{-\cancel{6x}\,3\,dt}{t\,\cancel{2x}}+\frac{1}{2}\operatorname{arctg}\frac{x}{2} =$$

$t=4+x^2$
$dt=2x\,dx$
$dx=\frac{dt}{2x}$

$$= \boxed{\frac{x^2}{2}-3\ln|4+x^2|+\frac{1}{2}\operatorname{arctg}\frac{x}{2}+C_1}$$

$$\int\frac{3x+2}{x^2-1}\,dx$$

$$\frac{3x+2}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1}=\frac{Ax+A+Bx-B}{x^2-1}$$

$x^1: 3=A+B$
$x^0: 2=A-B \quad)+$

$5=2A$
$A=\frac{5}{2} \quad B=\frac{1}{2}$

$$\int\frac{\frac{5}{2}}{x-1}dx+\int\frac{\frac{1}{2}}{x+1}dx = \boxed{\frac{5}{2}\ln|x-1|+\frac{1}{2}\ln|x+1|+C_2}$$

$$= \boxed{\frac{x^2}{2}-3\ln|4+x^2|+\frac{1}{2}\operatorname{arctg}\frac{x}{2}+\frac{5}{2}\ln|x-1|+\frac{1}{2}\ln|x+1|+C}$$

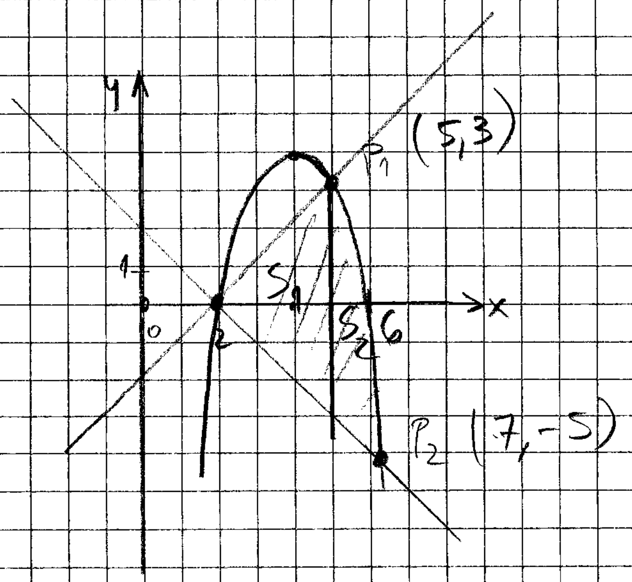
⑧ Izračunajte ploščino območja, ki ga omejujejo krivulje :

$y = -(x-2)(x-6)$

$y = x-2$

$\underline{y = -x+2}$

$T(4,4)$



$\boxed{y = x-2}$

$-(x-2)(x-6) = x-2$

$-(x-2)(x-6+1) = 0$

$(x-2)(x-5) = 0$

$\boxed{y = -x+2}$

$-(x-2)(x-6) = -(x-2)$

$-(x-2)(x-6-1) = 0$

$(x-2)(x-7)$

$S = S_1 + S_2$

$S_1 = \underbrace{3 \cdot 3 \cdot \frac{1}{2} \cdot 2 = 9}_{S_\triangle}$

$$S_2 = \int_5^7 \Big(\underbrace{-(x-2)(x-6)}_{zg.\ kriv} - \underbrace{(-x+2)}_{sp.\ kriv.} \Big) dx$$

$$= -\frac{x^3}{3} + \frac{9x^2}{2} - 14x \Big|_5^7 = \left(-\frac{7^3}{3} + \frac{9 \cdot 7^2}{2} - 14 \cdot 7\right) - \left(-\frac{5^3}{3} + \frac{9 \cdot 5^2}{2} - 14 \cdot 5\right) =$$

$$= -\frac{343}{3} + \frac{441}{2} - 98 + \frac{125}{3} - \frac{225}{2} + 70$$

$$-\frac{218}{3} + \frac{216}{2}$$

$$-\frac{218}{3} + 108 - 28 =$$

⑨ Izračunajte ploščino lika, ki ga omejujejo parametrično podane krivulje $r(\varphi) = 2\sqrt{\sinh(\sin\varphi)\cos\varphi}$ in pozitivna dela koordinatnih osi

$$S = \frac{1}{2}\int_\alpha^\beta r^2 d\varphi = \frac{1}{2}\int_0^{\frac{\pi}{2}} 4\,sh(\sin\varphi)\cos\varphi\, d\varphi$$

$0 < \varphi < \frac{\pi}{2}$

$$\int sh(\sin\varphi)\cos\varphi\, d\varphi = \int sh(t)dt = ch(t) + C$$

$ch(\sin\varphi) + C$

$t = \sin\varphi$

$dt = \cos\varphi\, d\varphi$

$$= 2ch(\sin\varphi)\Big|_{0}^{\frac{\pi}{2}} = 2\left(ch\left(\sin\frac{\pi}{2}\right) - ch(\sin(0))\right) = \underline{2(ch(1) - ch(0))}$$

$$ch\,x = \frac{e^{x} + e^{-x}}{2} \qquad = 2\left(\frac{e + e^{-1}}{2} - 1\right) = e + e^{-1} - 2$$

$$ch\,0 = 1$$

$$ch\,1 = \frac{e + e^{-1}}{2}$$