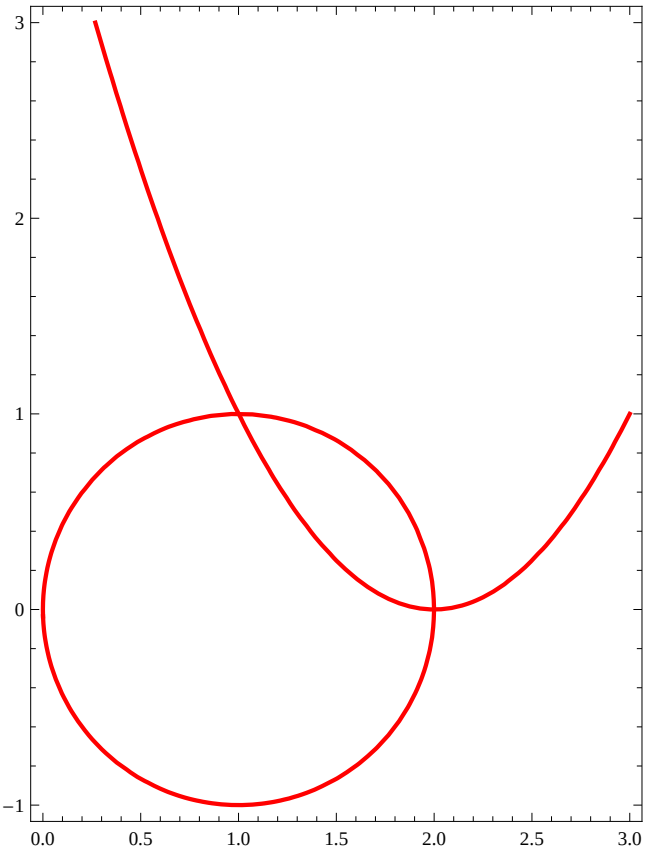
Dvojni integral

1. Narisi obmo■je D={(x,y); $(x-1)^2+y^2<1 \wedge y>(x-2)^2$ }
in izra■unaj dvojni integral

$$\iint_D x\,y\,\mathrm{d}x\,\mathrm{d}y$$

Rez: $\frac{41}{120}$

```
ContourPlot[{(x - 1)^2 + y^2 == 1, y == (x - 2)^2}, {x, 0, 3}, {y, -1, 3},
 AspectRatio → Automatic, ContourStyle → {{Red, Thick}, {Red, Thick}}]
```



Ukaz **Integrate**:
Nedolo■eni integral: **Integrate**[f,x]
Dolo■eni integral: **Integrate**[f,{x,xmin,xmax}]
Dvojni integral: **Integrate**[f,{x,xmin,xmax},{y,ymin,ymax}]

Naloga je resena na ve■ na■inov.
Prestudirajte vsako resitev s povdarkom na mejah integralov.

Najprej is■emo prese■is■a

```
Solve[{(x - 1)^2 + y^2 == 1, y == (x - 2)^2}, {x, y}]
```

$$\left\{\{x \to 1, y \to 1\}, \{x \to 2, y \to 0\},\ \left\{x \to \frac{1}{2}\left(5 - i\sqrt{7}\right), y \to \frac{1}{2}\left(-3 - i\sqrt{7}\right)\right\}, \left\{x \to \frac{1}{2}\left(5 + i\sqrt{7}\right), y \to \frac{1}{2}\left(-3 + i\sqrt{7}\right)\right\}\right\}$$

```
Solve[{(x - 1)^2 + y^2 == 1}, y]
Solve[{y == (x - 2)^2}, x]
```

$$\left\{\left\{y \to -\sqrt{2\,x - x^2}\right\}, \left\{y \to \sqrt{2\,x - x^2}\right\}\right\}$$

$$\left\{\left\{x \to 2 - \sqrt{y}\right\}, \left\{x \to 2 + \sqrt{y}\right\}\right\}$$

```
Integrate[x y, {x, 1, 2}, {y, (x - 2)^2, Sqrt[2 x - x^2]}]
```

$$\frac{41}{120}$$

```
Integrate[x y, {y, 0, 1}, {x, 2 - Sqrt[y], 1 + Sqrt[1 - y^2]}]
```

$$\frac{41}{120}$$

```
Integrate[Integrate[x y, {y, (x - 2)^2, Sqrt[2 x - x^2]}], {x, 1, 2}]
```

$$\frac{41}{120}$$

```
NIntegrate[Integrate[x y, {y, (x - 2)^2, Sqrt[2 x - x^2]}], {x, 1, 2}]
```

```
0.341667
```

2. Izra■unaj dvojni integral $\int\int_D e^{xy^2}\, y\, dx\, dy$,
obmo■je D je dolo■eno z ena■bami $y - 2x \le 0$, $2y - x \ge 0$, $xy \le 2$.
Obmo■je D tudi narisi!
Rez: 9.32233

```
Plot[{2 x , x / 2, 2 / x}, {x, 0, 4}, AspectRatio → Automatic, PlotStyle → Thick]
```

```
a = 2 x
b = x / 2
c = 2 / x
```

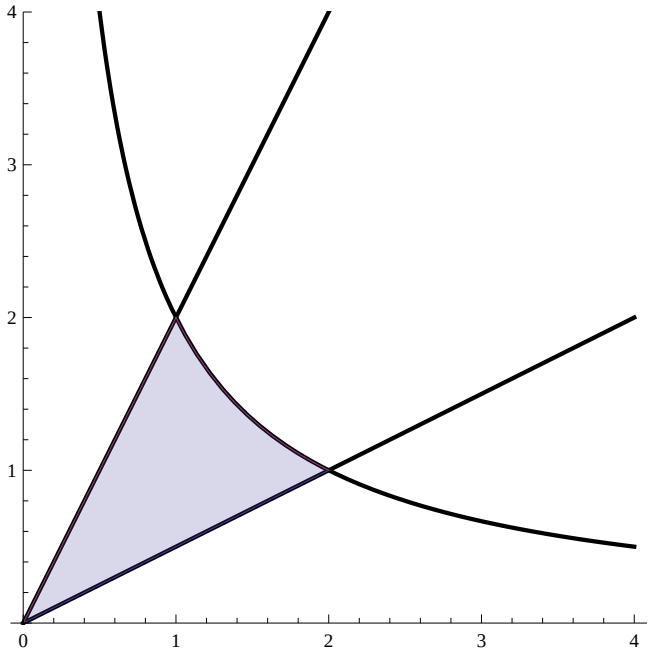
$$2\,x$$

$$\frac{x}{2}$$

$$\frac{2}{x}$$

Mnozico to■k v ravnini se da osen■iti z opcijo **Filling**, katere uporaba je bolj zahtevna. Prikazan je na■in osen■enja obmo■ja med dvema krivuljama.

```
Show[
 Plot[{2 x , x / 2, 2 / x}, {x, 0, 4}, AspectRatio → Automatic,
  PlotStyle → {{Thick, Black}}, PlotRange → {0, 4}],
 Plot[{x / 2, 2 x}, {x, 0, 1}, Filling → {1 → {2}}],
 Plot[{x / 2, 2 / x}, {x, 1, 2}, Filling → {1 → {2}}]
]
```




```
Solve[{y == 2 x, x y == 2}, {x, y}]
Solve[{y == 2 x, x  == 2 y}, {x, y}]
Solve[{x == 2 y, x y == 2}, {x, y}]
```

```
{{x → -1, y → -2}, {x → 1, y → 2}}
```

```
{{x → 0, y → 0}}
```

```
{{x → -2, y → -1}, {x → 2, y → 1}}
```

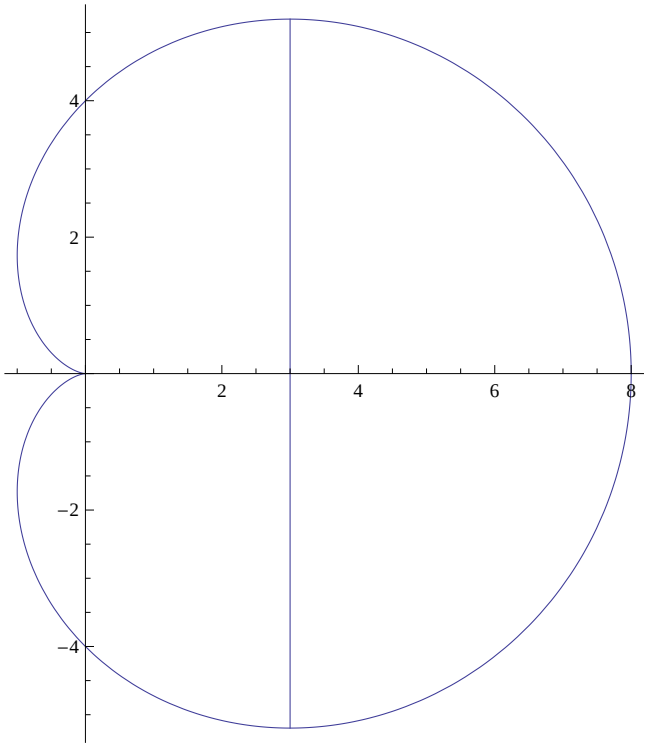
```
NIntegrate[E^(x y^2) y, {x,  0, 1}, {y, x / 2, 2 x}] +
 NIntegrate[E^(x y^2) y, {x, 1, 2}, {y, x / 2, 2 / x}]
```

```
9.32233
```

3. Izra■unaj plos■ino lika, ki ga dobis s presekom
$r \le 4\,(1 + \cos\varphi)$
$x \ge 3$
Rez: $9\sqrt{3} + 8\pi$

Krivuljo, ki je dana v polarnih kordinatah, se lahko narise z ukazom **PolarPlot**.

```
Krivulja1 = PolarPlot[4 (1 + Cos[φ]), {φ, 0, 2 π}];
Krivulja2 = PolarPlot[3 / Cos[φ], {φ, -π / 3, π / 3}];
Show[Krivulja1, Krivulja2]
```




```
Clear[r]
Solve[4 (1 + Cos[ϕ]) == 3 / Cos[ϕ], ϕ]
```

Solve::ifun : Inverse functions are being used by Solve, so

some solutions may not be found; use Reduce for complete solution information. ≫

$$\left\{\left\{\phi \to -\frac{\pi}{3}\right\}, \left\{\phi \to \frac{\pi}{3}\right\}, \left\{\phi \to -\text{ArcCos}\left[-\frac{3}{2}\right]\right\}, \left\{\phi \to \text{ArcCos}\left[-\frac{3}{2}\right]\right\}\right\}$$

```
Integrate[r, {ϕ, -Pi / 3, Pi / 3}, {r, 3 / Cos[ϕ], 4 (1 + Cos[ϕ])}]
```

$$9\sqrt{3} + 8\pi$$

4. Izra■unaj izlimitiran dvojni integral $\iint_D e^{-(x+y)^2}\, dx\, dy$

D: x ≥ 0, y ≥ 0,
tako da uvedes novi spremenljivki
x = r $\cos^2 t$
y = r $\sin^2 t$
Rez: 1/2

```
Clear[r, x, y]
x = r (Cos[t])^2;
y = r (Sin[t])^2;
J = Simplify[Det[{{D[x, r], D[x, t]}, {D[y, r], D[y, t]}}]]
Integrate[E^(-(r)^2) * J, {t, 0, Pi / 2}, {r, 0, Infinity}]
```

$$r \sin[2\,t]$$

$$\frac{1}{2}$$

5. Izra■unaj prostornino telesa, omejenega s ploskvami

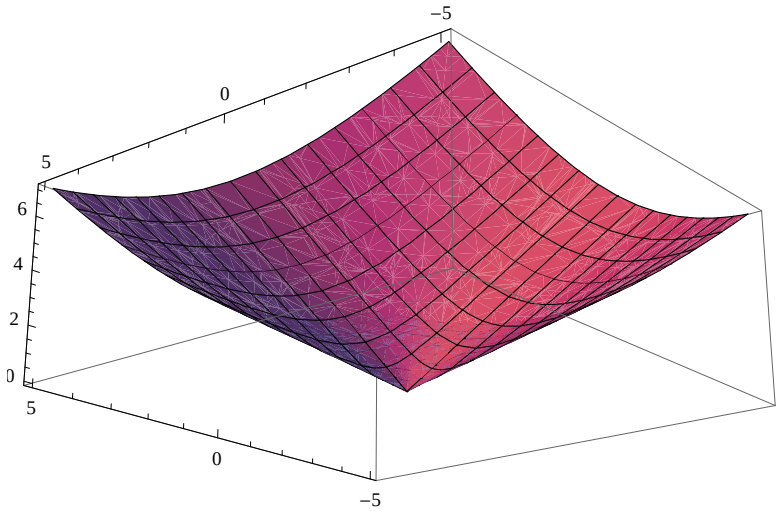
$z = \sqrt{x^2 + y^2}$

$x^2 + y^2 = 4\,x$

$z \geq 0$

Rez : 28.4444

```
Slika1 = Plot3D[Sqrt[x^2 + y^2], {x, -5, 5}, {y, -5, 5}]
```



Parametriziramo:

x = 2 Cos[u] + 2

y = 2 Sin[u]

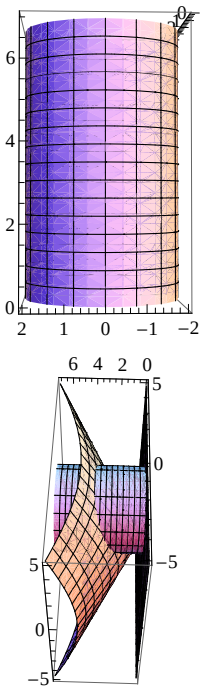
z = v

$$2 + 2\cos[u]$$

$$2\sin[u]$$

$$v$$

```
Slika2 = ParametricPlot3D[{2 Cos[u] + 2, 2 Sin[u], v}, {u, 0, 2 π}, {v, 0, 7}]
Slika3 = Plot3D[0, {x, -5, 5}, {y, -5, 5}];
Show[Slika1, Slika2, Slika3, ViewPoint -> {2.720, -2.012, -0.026}]
```



1. naloga

```
Clear[r]
Integrate[r * r, {ϕ, -Pi / 2, Pi / 2}, {r, 0, 4 Cos[ϕ]}]
```

$$\frac{256}{9}$$

2. naloga

```
Clear[r, x, y]
x = r Cos[ϕ] + 2
y = r Sin[ϕ]
NIntegrate[Sqrt[(r Cos[ϕ] + 2)^2 + (r Sin[ϕ])^2] * r, {ϕ, 0, 2 Pi}, {r, 0, 2}]
```

$2 + r \cos[\phi]$

$r \sin[\phi]$

Trojni integral

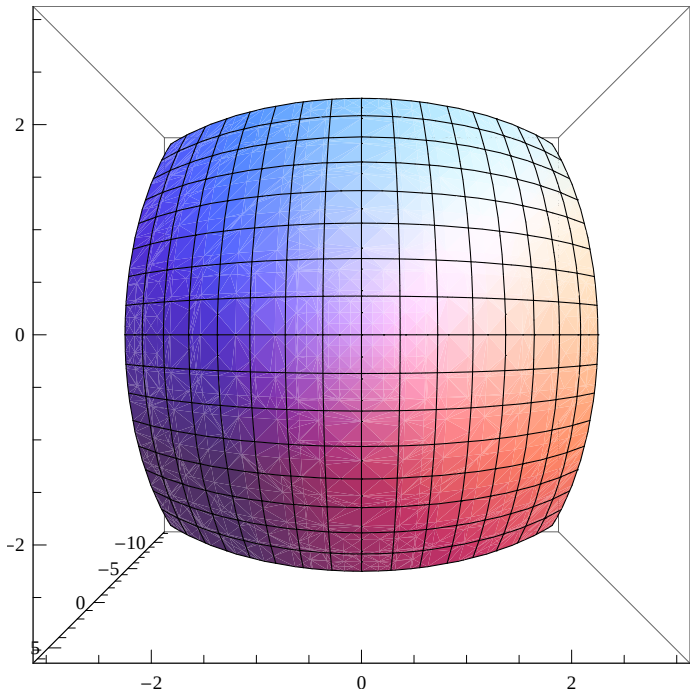
6. Izra■unaj prostornino telesa, omejenega z danima ploskvama:

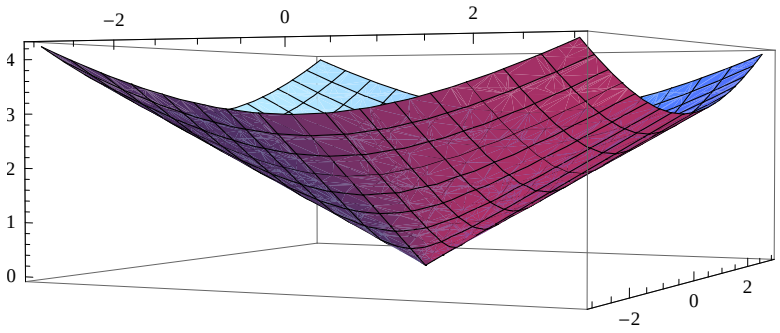
$z = 6 - x^2 - y^2$ in $z = \sqrt{x^2 + y^2}$

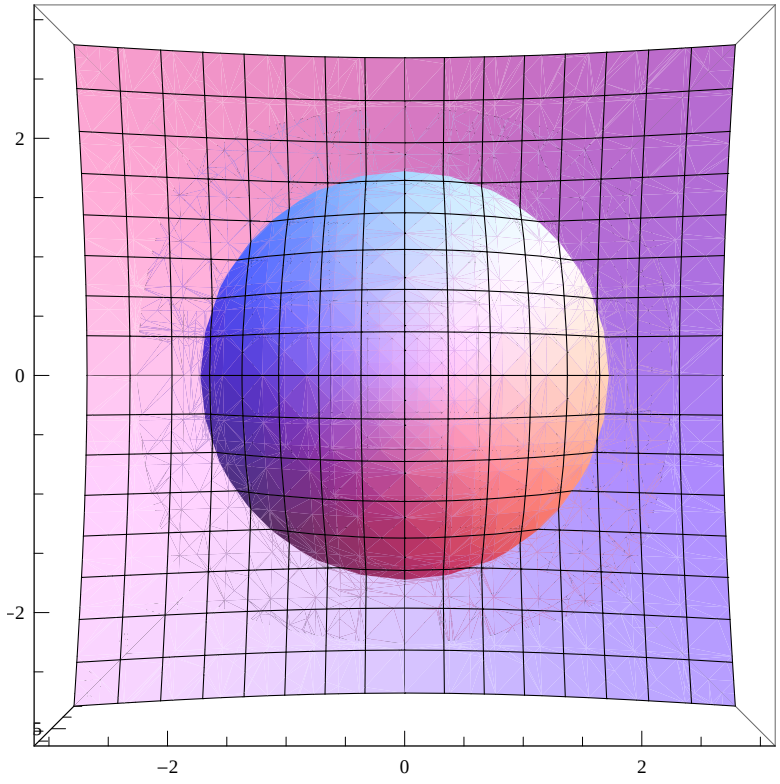
Navodilo: vpelji cilindri■ne kooedinate!

Rezultat: $\frac{32\pi}{3}$

```
Clear[x, y, z, S1, S2]
S1 = Plot3D[6 - x^2 - y^2, {x, -3, 3}, {y, -3, 3}, BoxRatios → {1, 1, 1}]
S2 = Plot3D[Sqrt[x^2 + y^2], {x, -3, 3}, {y, -3, 3}]
Show[S1, S2]
```




−2
0
2
4
3
2
1
0
−2
0
2


2
0
−2
−2
0
2

```
Integrate[r, {ϕ, 0, 2 Pi}, {r, 0, 2}, {z, r, 6 - r^2}]
```

$$\frac{32\,\pi}{3}$$

7. Dolo■i tezis■e 1/8 krogle $x^2 + y^2 + z^2 \le 1$ v prvem oktantu, ■e je gostota enaka $\rho = \frac{1}{\sqrt{1-(x^2+y^2+z^2)}}$.

Rezultat: $\left\{\frac{4}{3\,\pi}, \frac{4}{3\,\pi}, \frac{4}{3\,\pi}\right\}$

Formula za posamezno koordinato tezis■a se glasi

$$x_T = \frac{\iiint_V \rho x \, dx \, dx \, dz}{\iiint_V \rho \, dx \, dx \, dz}$$

Uvedi sferi■ne koordinate!

```
Clear[x, y, z, r, S1, S2]
ρ = 1 / Sqrt[1 - r^2]
x = Integrate[ρ r Cos[φ] Cos[θ] r^2 Cos[θ], {φ, 0, Pi / 2}, {θ, 0, Pi / 2}, {r, 0, 1}] /
  Integrate[ρ, {φ, 0, Pi / 2}, {θ, 0, Pi / 2}, {r, 0, 1}]
y = Integrate[ρ r Sin[φ] Cos[θ] r^2 Cos[θ], {φ, 0, Pi / 2}, {θ, 0, Pi / 2}, {r, 0, 1}] /
  Integrate[ρ, {φ, 0, Pi / 2}, {θ, 0, Pi / 2}, {r, 0, 1}]
z = Integrate[ρ r Sin[θ] r^2 Cos[θ], {φ, 0, Pi / 2}, {θ, 0, Pi / 2}, {r, 0, 1}] /
  Integrate[ρ, {φ, 0, Pi / 2}, {θ, 0, Pi / 2}, {r, 0, 1}]
```

$$\frac{1}{\sqrt{1 - r^2}}$$

$$\frac{4}{3\,\pi^2}$$

$$\frac{4}{3\,\pi^2}$$

$$\frac{4}{3\,\pi^2}$$

```
Tezisce = {x, y, z}
```

$$\left\{\frac{4}{3\,\pi^2}, \frac{4}{3\,\pi^2}, \frac{4}{3\,\pi^2}\right\}$$

```
ParametricPlot3D[{Cos[φ] Cos[θ], Sin[φ] Cos[θ], Sin[θ]},
 {φ, 0, Pi / 2}, {θ, 0, Pi / 2}, ViewPoint -> {5, 3, 4}]
```