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avditorne vaje naslednji teden

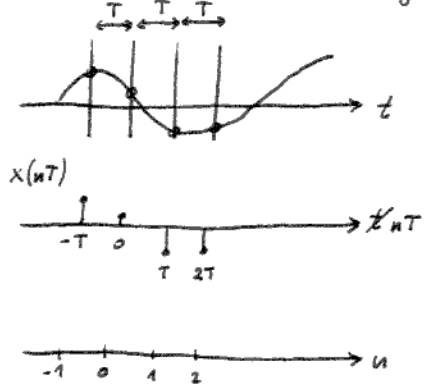
- domače naloge (4) (80%) (20%)
- vmesni testi (4) - zraven vsi zapiski (20%)
- 2 kolokvija (80%)
- ustni izpit

UVOD

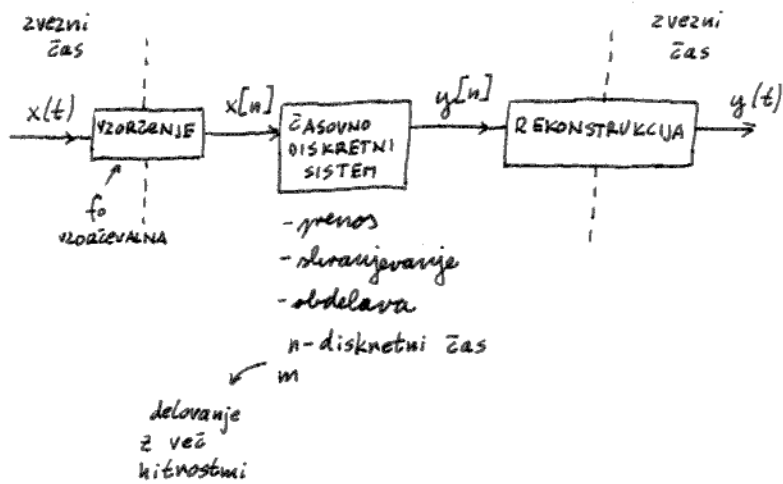
frekvenčna kretnica (en vhod, več izhodov, SIMO)



Vzorec → časovno diskreten signal  $x[n] = x(nT)$ ,  $T$  perioda vzorčenja

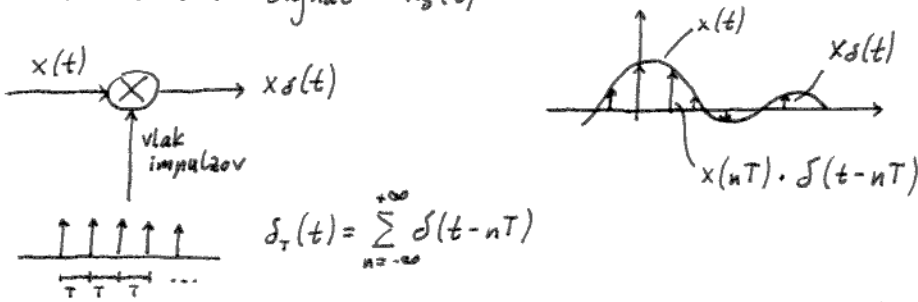


izguba informacije!



Vzorčenje in rekonstrukcija časovno zveznega signala

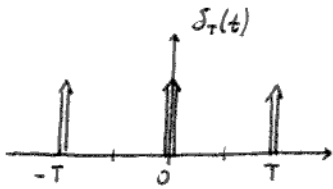
Idealno vzorčeni signal  $x_s(t)$



$$\delta_T(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

$$x_s(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT) = \sum_{n=-\infty}^{\infty} x[n] \delta(t - nT)$$

Razvoj vlaka impulzov v fourienovo vrsto:



$$X_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T}$$

Vsi koeficienti enaki

$$\omega_0 = \frac{2\pi}{T}$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{+jk\omega_0 t}$$

zmnožek:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT) = x(t) \cdot \delta_T(t) = x(t) \cdot \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{+jk\omega_0 t} = \frac{1}{T} \sum_{k=-\infty}^{\infty} x(t) e^{+jk\omega_0 t}$$

uporabimo lastnost FT o modulaciji:

$$x(t) \xleftrightarrow{FT} X(\omega)$$

$$x(t) e^{+jk\omega_0 t} \leftrightarrow X(\omega - k\omega_0)$$

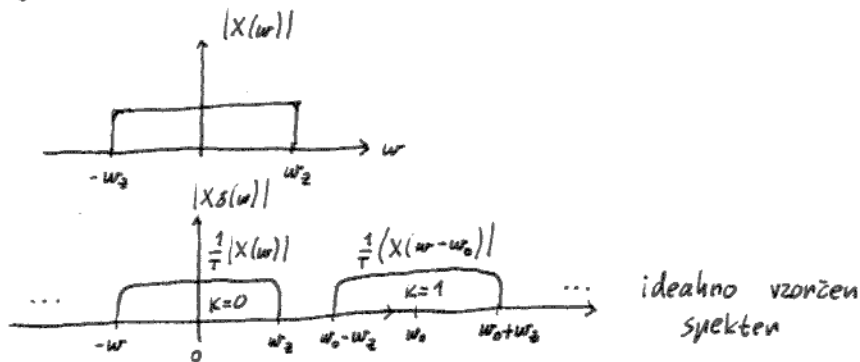
potem:

$$X_s(\omega) = F\{x_s(t)\} = F\left\{\frac{1}{T} \sum_{k=-\infty}^{\infty} x(t) e^{+jk\omega_0 t}\right\} = \frac{1}{T} \sum_{k=-\infty}^{\infty} F\{x(t) e^{+jk\omega_0 t}\} =$$

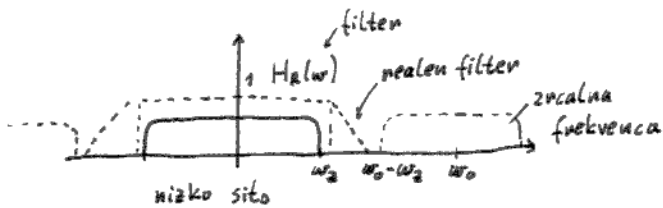
$$X_s(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_0)$$

spekter idealno vzorčenega signala

grafična predstavitev



vse kar je dveh odstranimo s filtrom  
(večkratniške frekvence vzorčenja  $w_0$ )



popolna rekonstrukcija zahteva:

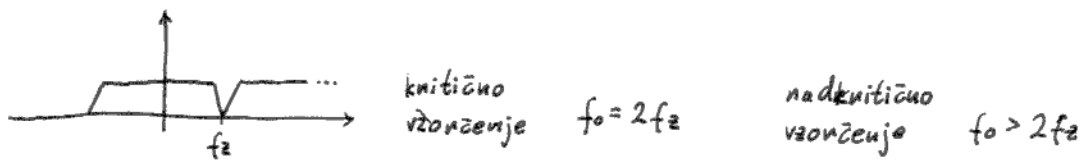
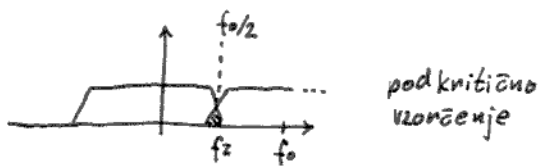
$$\begin{aligned} w_0 - w_2 &\geq w_2 \\ w_0 &\geq 2w_2 \end{aligned} \quad \left| \begin{aligned} f_0 &\geq 2f_z \end{aligned} \right|$$

zgornja meja  
frekvenca vzorčenega  
signala

Teorem o vzorčenju, 1949

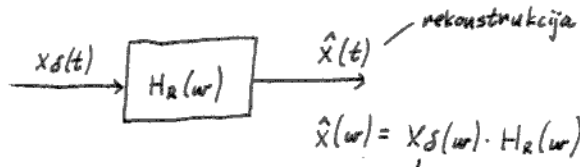
pripisuje se Shannonu  
začetnika Whittaken in Kotelnikov

Časovno zvezen signal je popolnoma rekonstruiran iz časovno diskretnega signala  $x[n]$ , če je  $x(t)$  frekvenčno omejen, frekvenčno vzorčenja na višja od  $2 \cdot f_z$ . Sicer pride do prekrivanja (aliasing)



Rekonstrukcija kritično vzorčenega signala

nek filter  $H_R(\omega) = \begin{cases} T, & \text{za } |\omega| < \frac{\omega_0}{2} \\ 0, & \text{za } |\omega| > \frac{\omega_0}{2} \end{cases}$   
 ↑  
 Rekonstrukcijski



$$\hat{x}(\omega) = X_S(\omega) \cdot H_R(\omega)$$

$$x_S(t) = \sum_{n=-\infty}^{\infty} x[n] \delta(t-nT)$$

$$X_S(\omega) = \int_{-\infty}^{\infty} x_S(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n] \delta(t-nT) e^{-j\omega t} dt = \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t-nT) e^{-j\omega t} dt = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega nT}$$

$$\hat{x}(t) = F^{-1} \left\{ X_S(\omega) \cdot H_R(\omega) \right\} = \frac{1}{2\pi} \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} X_S(\omega) H_R(\omega) e^{j\omega t} d\omega =$$

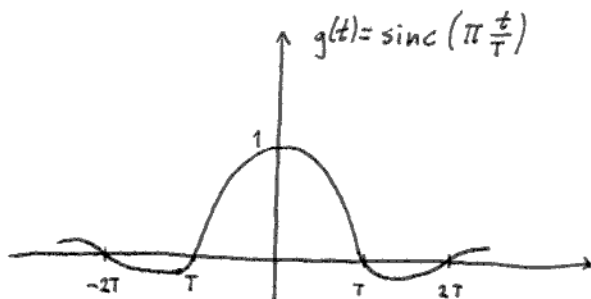
↑  
stojnična funkcija

$$= \frac{T}{2\pi} \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega nT} e^{j\omega t} d\omega = \sum_{n=-\infty}^{\infty} x[n] \frac{T}{2\pi} \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} e^{j\omega(t-nT)} d\omega =$$

$$= \sum_{n=-\infty}^{\infty} x[n] \frac{T (e^{j\frac{\omega_0}{2}(t-nT)} - e^{-j\frac{\omega_0}{2}(t-nT)})}{2\pi j(t-nT)} = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\frac{\omega_0}{2}(t-nT))}{\pi(\frac{t}{T}-n)} =$$

kritično vzorčen  $\omega_0 = \frac{2\pi}{T}$

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x[n] \underbrace{\frac{\sin(\pi(\frac{t}{T}-n))}{\pi(\frac{t}{T}-n)}}_{g(t)}$$



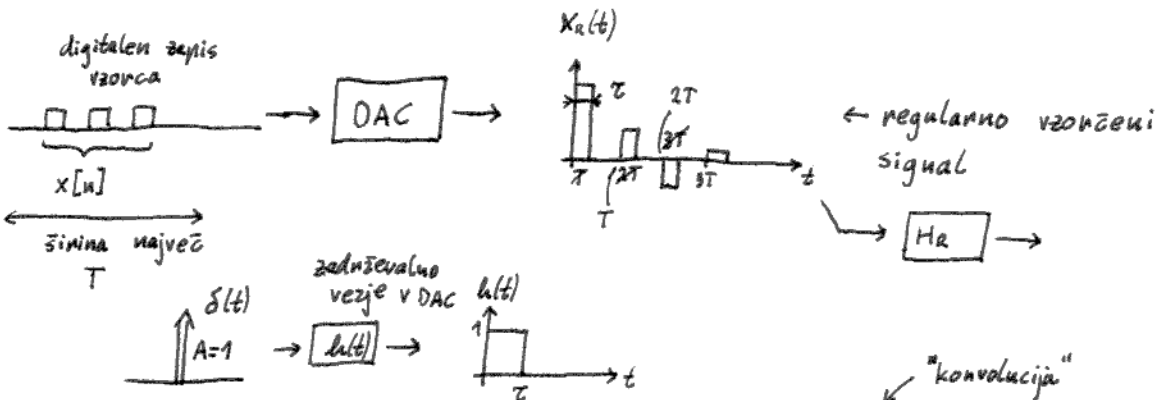
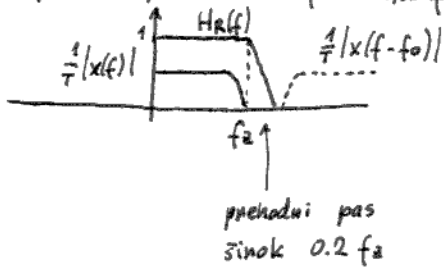
$$\hat{x}(t) = x_S(t) * g(t)$$



posledica pred vzrokom (idealni filter)

Rekonstrukcija nadkritično vzorčenega signala

v praksi ponavadi  $f_0 \geq 2,2 f_z$



$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n] \delta(t-nT) \quad x_R(t) = \sum_{n=-\infty}^{\infty} x[n] h(t-nT)$$

slika nepravilno vzorčenega signala

spekten  $x_R(t) = x_\delta(t) * h(t)$

$$X_R(\omega) = X_\delta(\omega) \cdot H_T(\omega)$$

$$H_T(\omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt = \int_0^T e^{-j\omega t} dt = T \frac{\sin(\frac{\omega T}{2})}{(\frac{\omega T}{2})} e^{-j\frac{\omega T}{2}}$$

amplituda    faza

filter:

$$X_{Rf}(\omega) = \frac{T}{T} \frac{\sin(\frac{\omega T}{2})}{(\frac{\omega T}{2})} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_0) e^{-j\frac{\omega T}{2}}$$

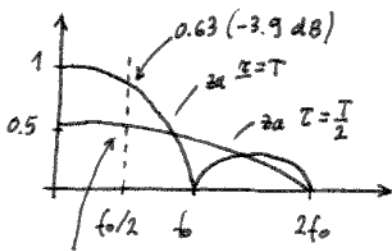
$$H_R(\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 - \omega \end{cases} \quad \text{filter za nadkritičnega}$$

spekten rekonstruiranega (filtriranega) nepravilno vzorčenega signala

$$\hat{X}_R(\omega) = \frac{T}{T} \text{sinc}\left(\frac{\omega T}{2}\right) X(\omega) e^{-j\frac{\omega T}{2}}$$

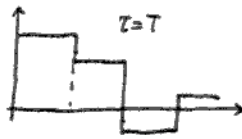
ojačanje    rahla frekvenčna odvisnost    spekten analognega    zakasnilen za  $\frac{T}{2}$

$$|H_2(\omega)| = \frac{T}{\tau} \text{sinc}\left(\frac{\omega\tau}{2}\right)$$



manjše  
popaženja  
pri mejni  
frekvenci

$$\sin\left(\frac{\omega_0 T}{2}\right) =$$



$$\sin\left(\frac{\omega T}{2}\right) = 0 \rightarrow \frac{\omega T}{2} = k\pi, k \neq 0$$

$$T = T \quad \omega = \frac{2\pi}{T} \cdot k$$

$$\omega = \frac{2\pi}{T} \cdot k = k \cdot \omega_0$$

kako izboljšati rekonstrukcijo?

↙ sodobna rešitev

1. digitalna interpolacija (dvig frekvence vzorčenja) niža rekonstrukcije se pomakne →
2. dovolj ozek impulz ( $\tau < \frac{1}{16}T$ ) ojača sicer pade za 16 → dodatno ojačanje
3. namesto konstantnega prepustnega pasu:  $H_k(\omega) = \frac{T}{\tau} \frac{\text{sinc}\left(\frac{\omega\tau}{2}\right)}{\text{sinc}\left(\frac{\omega T}{2}\right)}$  korekcijsko sito

### 3. Vrste diskretnih signalov

- PERIODIČNI
- APERIODIČNI
- NAKLJUČNI

poleg tega poznamo še:

- MOŽNOSTNI
- ENERGIJSKI

#### 3.1 Periodični signali

$$x[n] = x[n+N] \quad \text{za } n \in \mathbb{Z} \text{ in } N \in \mathbb{N}$$

diskreten signal dobljen z vzorčenjem analognega periodičnega signala v splošnem ni periodičen!

$$x(t) = x(t+T_A) = x\left(t+kT_A\right), k \in \mathbb{N}$$

$$x[n] = x(nT) = x\left(nT+kT_A\right) = x\left(\left(n+k\frac{T_A}{T}\right)T\right) \equiv$$

$$\equiv x\left[n+k\frac{T_A}{T}\right]$$

potem

$$\frac{f_0}{f_a} = \frac{N}{k} \quad \text{racionalno število}$$

$$N = k \frac{T_A}{T} = k \frac{f_0}{f_a} \cdot \frac{1}{T}$$

Zgled

$$x(t) = \sin(2\pi \cdot 1020 \cdot t)$$

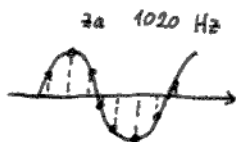
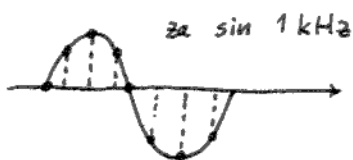
$$f_0 = 8 \text{ kHz}$$

$$f_a = 1020 \text{ Hz}$$

$$\frac{f_0}{f_a} = \frac{8000}{1020} = \frac{800}{102} = \frac{400}{51} \begin{matrix} \nearrow N \\ \searrow K \end{matrix}$$

signal ima periodo 400

oz. 51 period  $T_A$



Moč signala

$$P_x = \overline{x^2[n]} = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n]$$

|  
število  
vzorcev  
na periodo

Srednja vrednost

$$\overline{x[n]} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \quad (\text{povprečje})$$

efektivna:

$$X_{ef} = \sqrt{\overline{x^2[n]}}$$

$\sigma_x$  izmenična ef. vrednost

če  $\overline{x[n]} = 0$  velja  $X_{ef} = \sigma_x$   
 $P_x = \sigma_x^2$

Varianca ali disperzija

$$D = \sigma_x^2 = \overline{(x[n] - \overline{x[n]})^2} = \frac{1}{N} \sum_{n=0}^{N-1} (x[n] - \overline{x[n]})^2$$

↑  
std.  
deviacija

$$\sigma_x^2 = \overline{(x^2[n] - 2x[n]\overline{x[n]} + \overline{x[n]}^2)} = \overline{x^2[n]} - 2\overline{x[n]\overline{x[n]}} + \overline{\overline{x[n]}^2} =$$

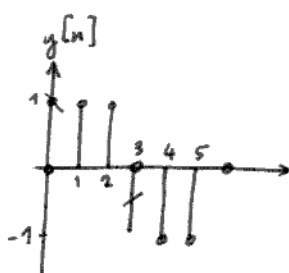
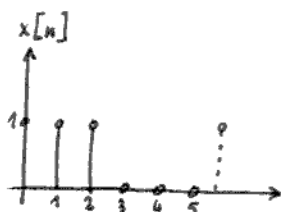
$$= \overline{x^2[n]} - 2\overline{x[n]}\overline{x[n]} + \overline{x[n]}^2 = \boxed{\overline{x^2[n]} - \overline{x[n]}^2}$$

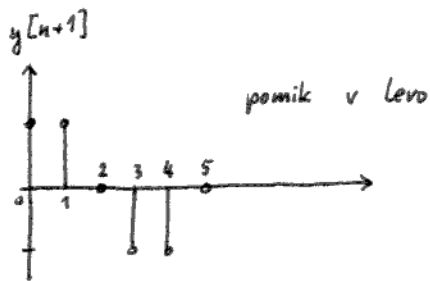
(  
srednja  
vrednost

Korelacija in avtokorelacija

duo signala  
enaka perioda

$$R_{xy}[m] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] y[n+m] \quad ; \quad m = 0, 1 \dots N-1$$





m	$R_{xy}[m]$
0	2/6
1	2/6
2	0
3	-2/6
4	-2/6
5	0



s korelacijo odkrivamo tudi fazno nazliko (maksimum korelacije pri prekrivanju)

Avto korelacija:

$$R_{xx}[m] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] x[n+m]; \quad m=0, 1, \dots, N-1$$

1.  $R_{xx}[m]$  je periodična s periodo  $N$

$$R_{xx}[m] = R_{xx}[m+N]$$

2.  $R_{xx}[0] \geq R_{xx}[m]$

3. sodost  $R_{xx}[m] = R_{xx}[-m]$

4.  $R_{xx}[0] = \overline{x^2[n]} = P_x$

### 3.2 Aperiodični signali

Energijski  $E_x < \infty$ ,  $P_x = 0$  čas opazovanja  $\rightarrow \infty$

Močnostni:  $0 < P_x < \infty$ ,  $E_x \rightarrow \infty$

Časovno omejeni signali s končno amplitudo so energijski:

$$E_x = \lim_{N \rightarrow \infty} \sum_{n=-N}^N x^2[n] < \infty \quad \begin{aligned} \overline{x[n]} &= 0 \\ \overline{x^2[n]} &= 0 \end{aligned} \Rightarrow \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N \begin{cases} x[n] \\ x^2[n] \end{cases}$$

$$r_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n] x[n+m] \quad \begin{aligned} &\text{dimenzija} \\ &\text{energije ...} \end{aligned}$$

Močnostni signali so časovno neomejeni, povprečna moč končna

$$P_x = \overline{x^2[n]} = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N x^2[n]$$

varianca

$$D = \overline{x^2[n]} - \overline{x[n]}^2 = \sigma_x^2$$

Diskretna konvolucija

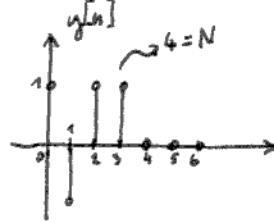
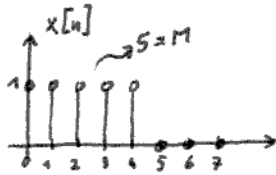
$$Z[n] = x[n] * y[n] = \sum_{m=-\infty}^{\infty} x[m] y[n-m]$$

$$x[n] * y[n] = y[n] * x[n] \text{ komutativnost}$$

4.3.2013

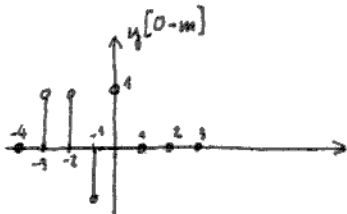
■ Zgled:

$$Z[n] = \sum_{m=-\infty}^{\infty} x[m] y[n-m] = x[n] * y[n]$$



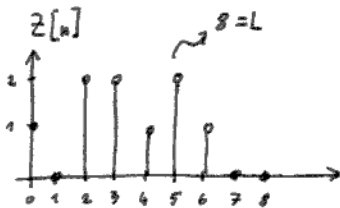
$x[m]$  enak!

$y[n-m]$  se pomika v desno, ko  $n$  raste?



sedaj napravimo vsoto in produkt

n	Z[n]
0	1
1	0
2	1
3	2
4	2
5	1
6	2
7	1
8	0
⋮	0

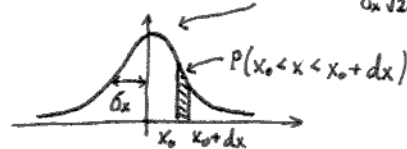


trajanje konvolucije:  
 $L = M + N - 1$

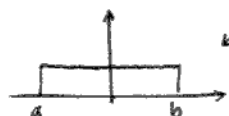
3.3 Naključni (diskretni) signali

Lahko so močnostni ali energijski signalov ne poznamo, poznamo le statistične parametre

- porazdelitev gostote verjetnosti (pdf)  $P_x(x)$
- gaussova ali normalna  $p_x(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x}{\sigma_x})^2}$



- laplaceova porazdelitev  $p_x(x) = \frac{1}{\sigma_x \sqrt{2}} e^{-\sqrt{2} \frac{|x|}{\sigma_x}}$



- enakomerna porazdelitev  $p_x(x) = \begin{cases} \frac{1}{b-a}, & \text{za } a < x < b \\ 0 & \text{sona} \end{cases}$

nadaljevanje statističnih parametrov:

- Avtokorelacija in gostota močnostnega spektra

### 3.4. Pomembnejši diskretni signali

a) Enotni impulz  $\delta[n]$

"Kroneckerjeva delta f"

$$\delta[n] = \begin{cases} 1 & \text{za } n=0 \\ 0 & \text{za } n \neq 0 \end{cases} \quad \begin{array}{l} \text{energija} \\ E_{\delta} = 1 \end{array}$$

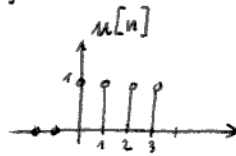
diracova:

$$E_{\delta} = \int_{-\infty}^{\infty} \delta^2(t) dt \rightarrow \infty$$

$\delta[n]$  je nevtralni element konvolucije  $\rightarrow x[n] * \delta[n] = x[n]$

b) Enotna stopnica  $u[n]$

$$u[n] = \begin{cases} 1 & \text{za } n \geq 0 \\ 0 & \text{za } n < 0 \end{cases}$$



diferenca:

$$\delta[n] = u[n] - u[n-1]$$

akumulacija:

$$u[n] = \sum_{m=-\infty}^n \delta[m]$$

c) Diskretni harmonični signal (realni)

$$x[n] = \cos(\Omega n)$$

periodičen signal, če velja  $\Omega = p \frac{2\pi}{N}$ ;  $p, N \in \mathbb{N}$

preverimo:

perioda je  $N$

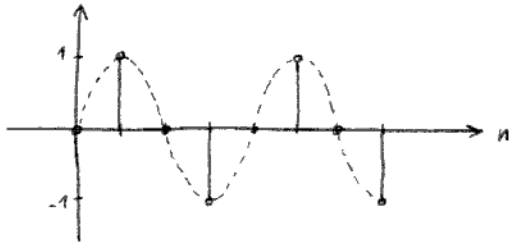
$$\cos\left(p \frac{2\pi}{N} (n+N)\right) = \cos\left(p \frac{2\pi}{N} n + p \frac{2\pi}{N} N\right) = \cos\left(p \frac{2\pi}{N} n\right)$$

1)  $\sin(\Omega_1 n) = \sin(\Omega_2 n)$  če velja  $\Omega_2 = \Omega_1 + k \cdot 2\pi$

2)  $\sin(\Omega_1 n) = -\sin(\Omega_2 n)$  če velja  $\Omega_2 = -\Omega_1 + k \cdot 2\pi$

$$x[n] = \sin\left(\frac{\pi}{2}n\right)$$

$$x[n] = \sin\left(\frac{6\pi}{2}n\right)$$



diskretno enaka spektra kljub višji frekvenci

## 4. Spektralna analiza diskretnih signalov

### 4.1 TDFT - časovno diskretna Fourierova transformacija

Definicija TDF ali DtFT

spekter signala določa FT:  $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$x[n] = \sum_{n=-\infty}^{\infty} x[n] \cdot \delta(t-nT) = X_S(t)$$

$$X_S(\omega) = \int_{-\infty}^{\infty} X_S(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n] \delta(t-nT) e^{-j\omega t} dt = \sum_{n=-\infty}^{\infty} x[n] \int_{-\infty}^{\infty} \delta(t-nT) e^{-j\omega t} dt =$$

sledi:

$$X_S(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega T n}$$

$$\Omega = \omega T = \frac{\omega}{f_0} = 2\pi \frac{f}{f_0}$$

↑  
normirana  
krožna  
frekvenca

$$\int_{-\infty}^{\infty} \delta(t-nT) e^{-j\omega t} dt = e^{-j\omega nT}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

TDFT

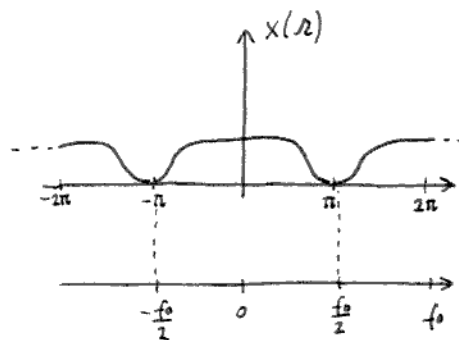
$$\int dt \rightarrow \sum$$

$$\omega \rightarrow \Omega$$

$$t \rightarrow n$$

$e^{-j\Omega}$  je periodična s periodo  $2\pi$

$$X(\Omega) = X(\Omega + k2\pi)$$



Inverzna TOFT

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega \quad \text{ITDFT}$$

### Lastnosti spektra $X(\Omega)$

$$X(\Omega) = |X(\Omega)| e^{j\phi(\Omega)} = A_x(\Omega) e^{j\phi_x(\Omega)}$$

tudi  
 $A_x(\Omega)$

$$\arg(X(\Omega)) = \phi_x(\Omega)$$

Za realne signale:  $x[n]$

$$X(-\Omega) = X^*(\Omega)$$

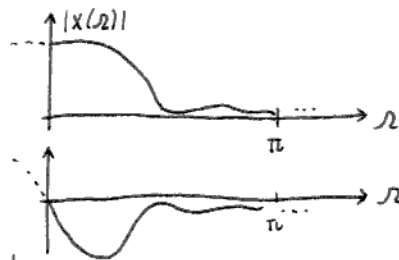
$$|X(-\Omega)| = |X(\Omega)| \quad \text{sodost}$$

$$\phi_x(-\Omega) = -\phi_x(\Omega) \quad \text{Lihost}$$

Spektre podajamo na intervalu:

$$0 < \Omega < \pi$$

$$0 < f < \frac{f_0}{2}$$



Za sode signale je  $X(\Omega)$  realen!

Za lihe signale je  $X(\Omega)$  čisto imaginaren!

### Lastnosti transformacije (preslikava)

$$x_i[n] \leftrightarrow X_i(\Omega)$$

1. Linearnost:

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow a X_1(\Omega) + b X_2(\Omega)$$

2. Časovni premik:

$$x[n-n_0] \leftrightarrow X(\Omega) \cdot e^{-j\Omega n_0}$$

3. Premik spektra:

$$x[n] e^{j\Omega_0 n} \leftrightarrow X(\Omega - \Omega_0)$$

$$\sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega_0 n} e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j(\Omega - \Omega_0)n} = X(\Omega - \Omega_0)$$

## 4. Modulacija

$$x[n] \cdot \cos(\Omega_0 n) = \frac{1}{2} x[n] e^{j\Omega_0 n} + \frac{1}{2} x[n] e^{-j\Omega_0 n}$$

$$x[n] \cdot \cos(\Omega_0 n) \leftrightarrow \frac{1}{2} X(\Omega - \Omega_0) + \frac{1}{2} X(\Omega + \Omega_0)$$

## 5. Konvolucija

$$x_1[n] * x_2[n] \leftrightarrow X_1(\Omega) \cdot X_2(\Omega)$$

Dokaz:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} (x_1[n] * x_2[n]) e^{-j\Omega n} &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (x_1[m] x_2[n-m]) e^{-j\Omega n} = \sum_{m=-\infty}^{\infty} x_1[m] \sum_{n=-\infty}^{\infty} x_2[n-m] e^{-j\Omega n} = \\ &= \sum_{m=-\infty}^{\infty} x_1[m] \underbrace{\left( \sum_{k=-\infty}^{\infty} x_2[k] e^{-j\Omega k} \right)}_{X_2(\Omega)} e^{-j\Omega m} = X_2(\Omega) \underbrace{\sum_{m=-\infty}^{\infty} x_1[m] e^{-j\Omega m}}_{X_1(\Omega)} \quad \text{qed} \end{aligned}$$

$\downarrow$   
 $n-m=k$   
 $n=k+m$

## 6. Množenje oz. spekten produkta

11.3.2013

$$X_1[n] \cdot X_2[n] \leftrightarrow \frac{1}{2\pi} X_1(\Omega) * X_2(\Omega)$$

Dokaz:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} x_1[n] x_2[n] e^{-j\Omega n} &= \sum_{n=-\infty}^{\infty} \left( \frac{1}{2\pi} \int_0^{2\pi} X_1(\Omega) e^{j\Omega n} d\Omega \right) \cdot x_2[n] \cdot e^{-j\Omega n} = \\ &= \frac{1}{2\pi} \int_0^{2\pi} X_1(\Omega) \sum_{n=-\infty}^{\infty} x_2[n] \underbrace{e^{-j\Omega n} e^{j\Omega n}}_{e^{-j(\Omega-\Omega)n}} d\Omega = \frac{1}{2\pi} \int_0^{2\pi} X_1(\Omega) X_2(\Omega - \Omega) d\Omega = \\ &= \frac{1}{2\pi} X_1(\Omega) * X_2(\Omega) \end{aligned}$$

$\checkmark$  konvolucija po frekvenci (cirkularna)

za dobne ločljivosti rabimo dolgo meritev

## 7. Konjugacija

$$X^*[n] \leftrightarrow X^*(-\Omega)$$

za realne signale  $X^*[n] = x[n]$ 

$$X^*(-\Omega) = X(\Omega)$$

8. Avtokorelacija

$$r_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n] \cdot x[n+m] \quad \text{Energijski signali}$$

$$r_{xx}[m] \leftrightarrow X(\omega) \cdot X^*(\omega) = W_x(\omega)$$

↑  
energijski  
spekter, realna vrednost

$$R_{xx}[m] = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N x[n] \cdot x[n+m] \quad \text{Možnostni signali}$$

$$R_{xx}[m] \leftrightarrow \lim_{N \rightarrow \infty} \left( \frac{1}{2N+1} X(\omega) \cdot X^*(\omega) \right) = \lim_{N \rightarrow \infty} \left( \frac{1}{2N+1} W(\omega) \right) = S_x(\omega)$$

gostota možnostenega  
energijskega  
spektra (energija na  
diferencialni zivni  
spektra)

9. Parsevalov teorem

Energijski signali

$$r_{xx}[m] = \frac{1}{2\pi} \int_0^{2\pi} W_x(\omega) e^{j\omega m} d\omega$$

za  $m=0$ :

$$\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi} \int_0^{2\pi} W_x(\omega) d\omega$$

Možnostni signali

$$R_{xx}[m] = \frac{1}{2\pi} \int_0^{2\pi} S_x(\omega) e^{j\omega m} d\omega$$

$$P_x = R_{xx}[0] = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N x^2[n] = \frac{1}{2\pi} \int_0^{2\pi} S_x(\omega) d\omega$$

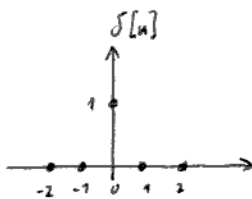
za  $f = 2\pi$  odpade  
 $\omega = 2\pi f \rightarrow f = \frac{\omega}{2\pi}$

TDFT Nekaterih signalov

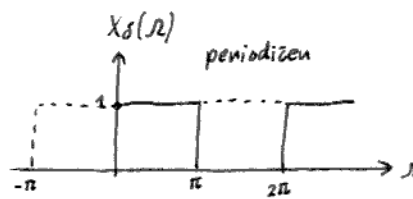
1. Enotin impulz

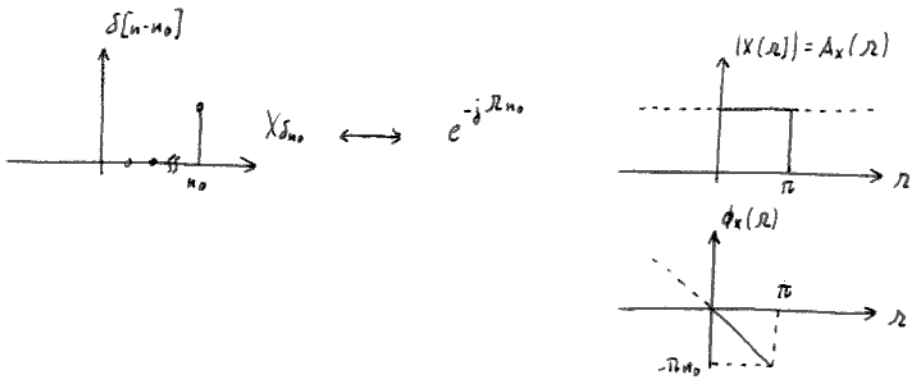
$$x[n] = \delta[n]$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$$



↔

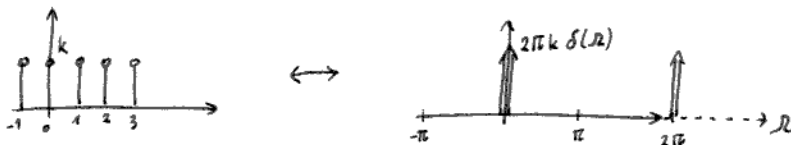




2. Konstanten signal

$x[n] = k \leftrightarrow X_k(\omega) = 2\pi k \delta(\omega)$  za  $-\pi < \omega < \pi$

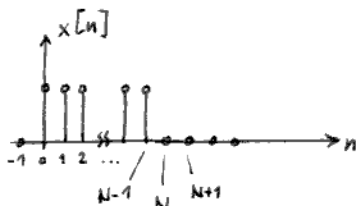
↖ zvezna diracova fja



$$X[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi k \delta(\omega) e^{j\omega n} d\omega = \frac{2\pi}{2\pi} k = k$$

3. Pravokotni signal

$x[n] = u[n] - u[n-N]$  ima  $N$  impulzov



$$X(\omega) = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} = \frac{e^{j\omega \frac{N}{2}} - e^{-j\omega \frac{N}{2}}}{e^{j\omega \frac{1}{2}} - e^{-j\omega \frac{1}{2}}} \cdot \frac{e^{-j\omega \frac{N}{2}}}{e^{-j\omega \frac{1}{2}}} =$$

$$\sum_{n=0}^{N-1} a^n = 1 + a + a^2 + \dots + a^{N-1} \cdot \frac{1-a}{1-a} = \frac{1-a^N}{1-a} \quad \left| \quad = \frac{\sin(\frac{\omega N}{2})}{\sin(\frac{\omega}{2})} \cdot e^{-j\omega \frac{N-1}{2}} =$$

$X(\omega) = \lim_{N \rightarrow \infty} X(\omega) =$

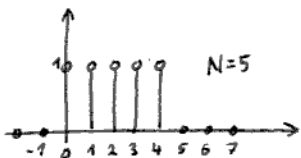
$= \lim \frac{\cos(\frac{\omega N}{2}) \frac{N}{2}}{\cos(\frac{\omega}{2}) \cdot \frac{1}{2}} = N$

$|X(\omega)| = 0; \sin(\frac{\omega N}{2}) = 0 \Rightarrow \frac{\omega N}{2} = k\pi; k \neq 0$

$\omega = \frac{k}{N} \cdot 2\pi; k \neq m \cdot N$

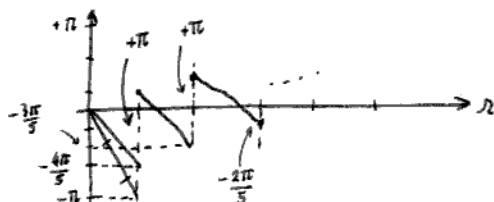
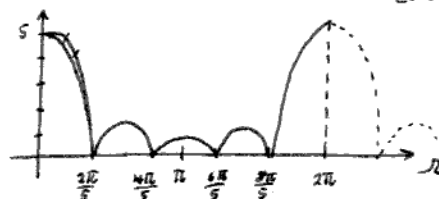
↓ celo stevilo...

nek primer:



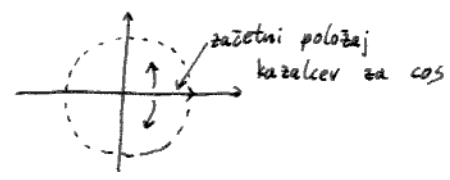
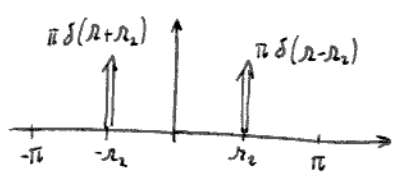
$X(\omega) = \frac{\sin(\frac{5}{2}\omega)}{\sin(\frac{\omega}{2})} e^{-j\omega \frac{4}{2}}$

↖ položaj srednjega signala pulza (če sodi signal eksponenta ni)



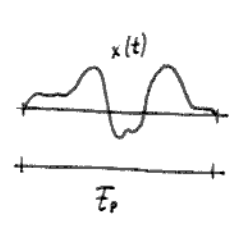


$$x[n] = \cos(\Omega_2 n) \leftrightarrow \frac{1}{2} \pi \delta(\Omega - \Omega_2) + \frac{1}{2} \pi \delta(\Omega + \Omega_2)$$



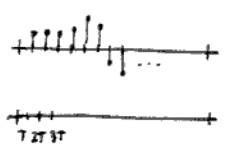
## 4.2 Diskretna Fournierjeva transformacija (DFT)

Definicija:



$$x(t) = x(t + T_p)$$

$$\omega_p = \frac{2\pi}{T_p}$$



osnovna frekv.  
 $f_p = \frac{1}{T_p}$   
 $f_0 \geq 2 \cdot k_m \cdot f_p$   
 $N \geq 2 \cdot k_m$

↑  
 toliko vzorcev v periodi  $T_p$

$$x(t) = \sum_{k=-k_m}^{+k_m} X_k e^{jk\omega_p t}$$

Fournierova vrsta

omejeno število komponent  
 $k_m$  najvišja herm. komp.

$$X_k = \frac{1}{T_p} \int_0^{T_p} x(t) e^{-jk\omega_p t} dt$$

$$X_k = \frac{1}{NT} \sum_{n=0}^{N-1} x[n] e^{-jk \frac{2\pi}{NT} n \cdot T} \cdot T = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk \frac{2\pi}{N} n}$$

↓  
 kot dt  
 /  
 periode vzorčenja

rezultat potem:

$$X_k \rightarrow X[k]$$

18.3.2013

preslikava n vzorcev (čas) v frekvenčni prostor

$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} k \cdot n}$$

↑  
 kdaj tudi  $\frac{1}{NT}$

DFT  $X[k]$  je periodičen spekter s periodo N  
 k omejimo od 0 do N-1

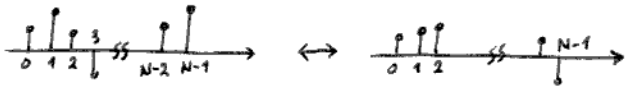
inverzna:

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi}{N} k \cdot n}$$

IDFT n med 0 in N-1, postane periodična ...

$$x[n] \leftrightarrow X[k]$$

DFT

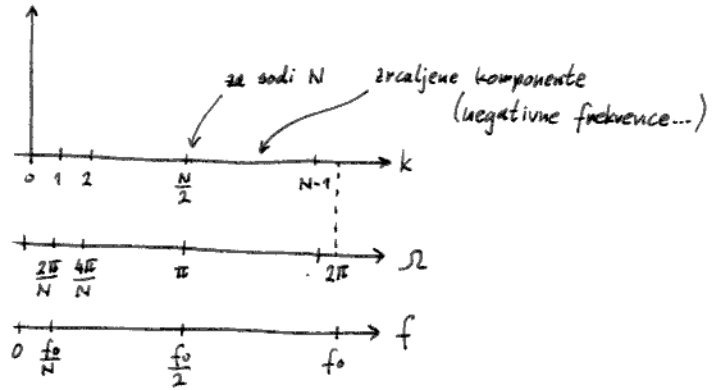


Lastnosti spektra X[k]

$$X[k] = |X[k]| \cdot e^{j\phi_x[k]} = A_x[k] e^{j\phi_x[k]}$$

$$\Omega_k = \frac{2\pi f}{N} \cdot k = \frac{\omega_k}{f_0} = 2\pi \frac{f_k}{f_0}$$

$f_k = \frac{f_0}{N} \cdot k$



Lastnosti zaradi periodičnosti:

$$X[k] = X[k+N]$$

$$X[-k] = X[N-k]$$

za realne signale:

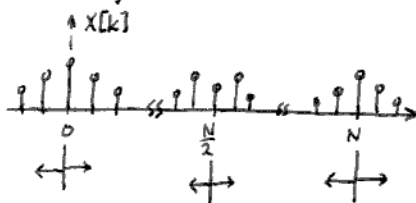
$$X[-k] = X^*[k] \rightarrow \frac{1}{N} \sum_{n=0}^{N-1} x^*[n] e^{j\frac{2\pi}{N}k \cdot n} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}(-k)n}$$

$$X^*[k] = X[N-k]$$

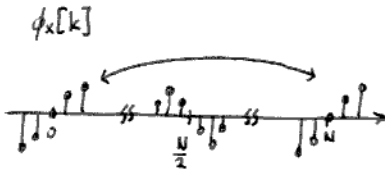
$$|X[-k]| = |X[k]| \quad \text{faza je liha} \quad \phi_x[-k] = -\phi_x[k]$$

$$|X[k]| = |X[N-k]| \quad \text{faza} \quad \phi_x[k] = -\phi_x[N-k]$$

simetrija amplitude:



asimetrije faze:



za realne  $x[n]$  zadošča da poznamo  $X[0]$  do  $X[\frac{N}{2}]$   
 oz  $X[\frac{N-1}{2}]$  ↑ sodi N  
 ↑ lihi N

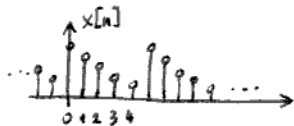
Lastnosti preslikave DFT

par  $x_i[n] \leftrightarrow X_i[k]$

1. Linearnost

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1[k] + bX_2[k]$$

2. Časovni premik

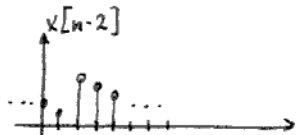


$$x[n-n_0] \leftrightarrow X[k] \cdot e^{-j\frac{2\pi}{N}k \cdot n_0}$$

$\rightarrow m \cdot \text{mod } N = m - L \cdot N$  da velja  $0 < m - L \cdot N < N$  in  $L \in \mathbb{Z}$

$$x[n-n_0] \rightsquigarrow = X[(n-n_0) \text{ mod } N]$$

$2 \text{ mod } 5 = 2$   
 $-2 \text{ mod } 5 = 3$



ostanek po celoštevilskem deljenju

3. Premik spektra

$$x[n] \cdot e^{jk_0 n} \leftrightarrow X[k-k_0]$$

zaradi periodičnosti velja  
 $e^{jk_0 n} = e^{j(k_0 \text{ mod } N)n}$

$$k_0 \Omega_0 = k_0 \frac{2\pi}{N}$$

"ločljivost"

4. Modulacija

$$x[n] \cdot \cos(k_0 n) \leftrightarrow \frac{1}{2}(X[k-k_0]) + \frac{1}{2}(X[k+k_0])$$

5. Konvolucija

$x_1[n]$  in  $x_2[n]$  periodična z enakima periodama  $N$

$$\text{krožna konvolucija } x_1[n] * x_2[n] = \sum_{m=0}^{N-1} x_1[m] x_2[n-m]$$

potem:

$$x_1[n] * x_2[n] \leftrightarrow N X_1[k] \cdot X_2[k]$$

ker v preslikavi  $\frac{1}{N}$

$$x_2[(n-m) \text{ mod } N] = \begin{cases} x_2(n-m+N) & \text{za } (n-m) < 0 \\ x_2(n-m) & \text{za } 0 < n-m < N \end{cases}$$

↑  
ciklični premik

6. Množenje

$$x_1[n] \cdot x_2[n] \leftrightarrow X_1[k] * X_2[k]$$

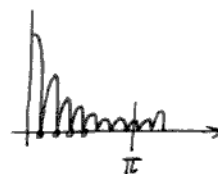
↑  
tudi oknenje

pravokotno oknenje nam dodaja harmonike

krožna konvolucija

$$\sum_{l=0}^{N-1} X_1[l] \cdot X_2[k-l]$$

spekter pravokotnega okna:



## 7. Avtokorelacija

(za periodične signale)

$$R_{xx}[m] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] x[n+m]$$

↑  
(n+m) mod N

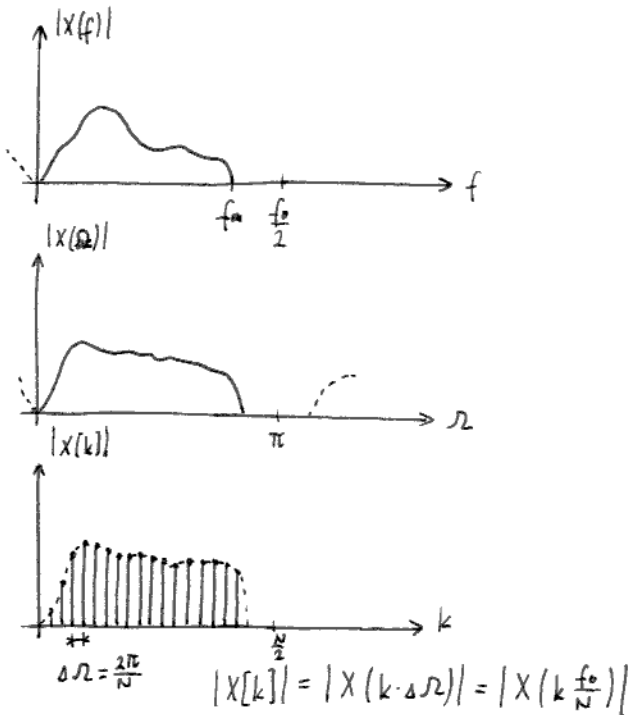
$$R_{xx}[m] \leftrightarrow \frac{1}{N} \sum_{m=0}^{N-1} R_{xx}[m] e^{-j \frac{2\pi}{N} k \cdot m} = |X[k]|^2$$

$$S_x[k] = |X[k]|^2 \quad \text{močnostni spekter}$$

## 8. Parsevalov teorem

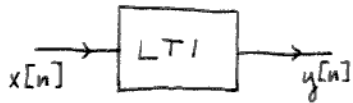
$$R_{xx}[m] = \text{IDFT} \left\{ |X[k]|^2 \right\} = \sum_{k=0}^{N-1} |X[k]|^2 e^{j \frac{2\pi}{N} k \cdot m}$$

$$R_{xx}[0] = P_x = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n] = \sum_{k=0}^{N-1} |X[k]|^2$$



# 5. Linearni diskretni sistemi

Definicije:



1. Linearnost (superpozicija)
2. Časovna nespremenljivost
3. Stabilnost
4. Kazualnost oz. vzročnost

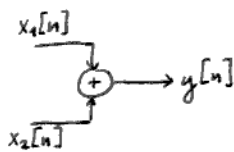
Stabilnost:

omejen vhod ↔ omejen izhod

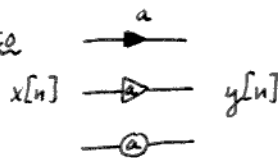
Sestava in opis sistemov:

Gradniki:

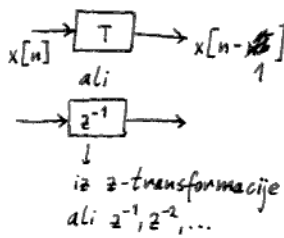
- seštevalnik



- množilnik s konstanto

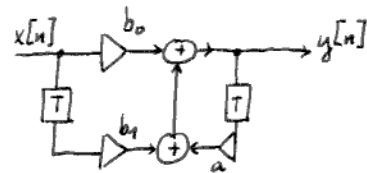


- zakasnilnik



Diferenčna enačba in shema diskretnega sistema:

■ Zyled:



$$y[n] = b_0 x[n] + b_1 x[n-1] + a y[n-1]$$

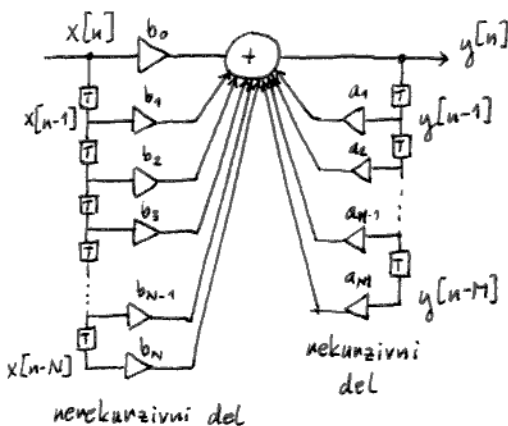
Posplošena diferenčna enačba:

$$y[n] = \sum_{i=0}^N \phi_i x[n-i] + \sum_{j=1}^M a_j y[n-j]$$

nerekurzivni del (direktni del)

rekurzivni del (povratna vezava)

Direktna struktura I:



- če so vsi  $a_j = 0$  je sistem nerekurziven

- če je vsaj en  $a_j \neq 0$  je sistem rekurziven

25.3.2013

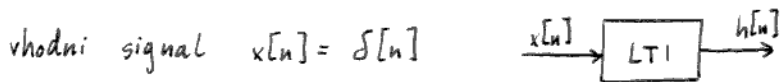
- opis sistema v časovnem prostoru z odzivom na enotni impulz
- opis sistema v frekvenčnem prostoru s frekvenčnim odzivom
- opis sistema s sistemsko funkcijo  $H(z)$  oz. legumi ničel in polov sist. fje.

Dodatek k FFT:

FFT lahko uporabimo če je dolžina niza enaka  $2^n$ !

lahko seveda umetno dodamo ničle in povežamo natančnost

### 5.3. Impulzni odziv $h[n]$



določitev izhodnega signala za poljubno vzbujanje:

- 1)  $\delta[n] \rightarrow h[n]$
- 2)  $\delta[n-i] \rightarrow h[n-i]$
- 3)  $x[i] \delta[n-i] \rightarrow x[i] h[n-i]$
- 4) upoštevamo vse zamike  $i$  in vse signale seštejemo

$$\sum_{i=-\infty}^{\infty} x[i] \delta[n-i] = x[n] \rightarrow \sum_{i=-\infty}^{\infty} x[i] h[n-i] = y[n] \quad \text{pravzaprav konvolucija}$$

$$y[n] = x[n] * h[n] = \sum_{i=-\infty}^{\infty} x[i] h[n-i] = \sum_{i=-\infty}^{\infty} h[i] x[n-i]$$

za desno-stranski odziv  $h[n] = 0$  za  $n < 0$ :

$$y[n] = \sum_{i=-\infty}^n x[i] h[n-i]$$

in če je  $x[n] \neq 0$  le za  $n \geq 0$ :

$$\boxed{y[n] = \sum_{i=0}^n x[i] h[n-i]}$$

za kausalne  
in desnostranske [odziv]  
vhodne signale

### Lastnosti LTI sistema na osnovi $h[n]$

sistem je stabilen, če velja  $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$  absolutna sumabilnost je potreben in zadosten pogoj

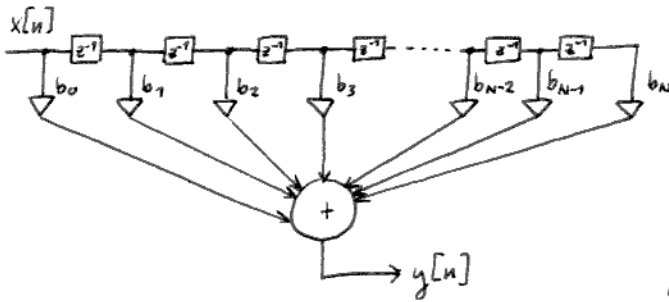
sistem je kavzalen, če velja  $h[n] = 0$  za vse  $n < 0$  in s tem hkrati ueneničljiv

glede na trajanje  $h[n]$  - števila neničelnih elementov  $h[n]$  ločimo dve skupini sistemov:

- a) FIR - končni impulzni odziv (finite impulse response)
- b) IIR - neskončni odziv (infinite ...)

a) FIR sistemi

$h[n]$  - končno število neničelnih vzorcev  
vsi nenekurzivni sistemi so FIR



$$h[n] = \sum_{i=0}^N b_i \delta[n-i]$$

$$h[n] = \begin{cases} b_n & \text{za } 0 \leq n \leq N \\ 0 & \text{drugod} \end{cases}$$

izhod potem:

$$y[n] = \sum_{i=0}^N b_i x[n-i]$$

b) IIR sistemi

teoretično trajna odziv neskončno dolgo  
za stabilne sisteme mora  $h[n]$  izveneti

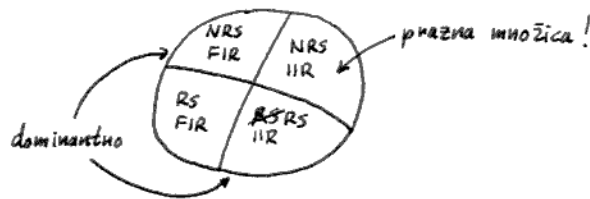
$$\sum_{n=0}^{\infty} |h[n]| < \infty \Rightarrow \lim_{n \rightarrow \infty} h[n] = 0$$

Zveza med strukturo in odzivom:

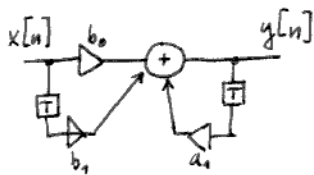
nenekurziven sistem  $\rightarrow$  FIR odziv  
IIR  $\rightarrow$  rekurziven sistem

- paščice veljajo v eno smer!

če to lepo razvidno narišemo:



■ Zgled: določitev  $h[n]$



$$y[n] = b_0 x[n] + b_1 x[n-1] + a_1 y[n-1]$$

izčemo  $h[n]$   
 $x[n] = \delta[n]$  in  $y[-1] = 0$  "mrtev" sistem

$$h[n] = b_0 \delta[n] + b_1 \delta[n-1] + a_1 h[n-1]$$

Rekurzivno reševanje:

$$h[0] = b_0 + b_1 \cdot 0 + a_1 \cdot 0 = b_0$$

$$h[1] = b_0 \cdot 0 + b_1 + a_1 \cdot b_0 = b_1 + a_1 b_0$$

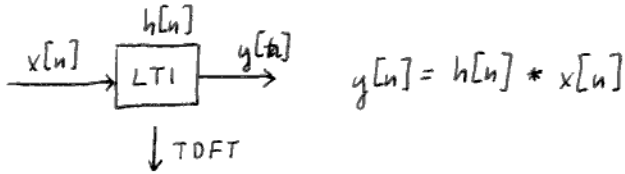
$$h[2] = a_1 h[1] = a_1 b_1 + a_1^2 b_0 = a_1 (b_1 + a_1 b_0)$$

$$h[3] = (b_1 + a_1 b_0) a_1^2$$

$$\vdots$$

$$h[n] = (b_1 + a_1 b_0) a_1^{n-1} = \left(\frac{b_1}{a_1} + b_0\right) a_1^n \Rightarrow h[n] = \begin{cases} 0, & \text{za } n < 0 \\ b_0, & \text{za } n = 0 \\ \left(\frac{b_1}{a_1} + b_0\right) a_1^n, & \text{za } n > 0 \end{cases}$$

5.4 Frekvenčni odziv LTI sistema

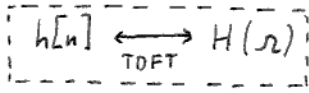


$$y[n] = h[n] * x[n]$$

$$Y(\Omega) = H(\Omega) \cdot X(\Omega)$$

$H(\Omega)$  je frekvenčni odziv

$$H(\Omega) = |H(\Omega)| e^{j\phi_H(\Omega)}$$



za realen  $h[n]$

$$H(-\Omega) = H^*(\Omega)$$

↑  
amplitudni odziv

↑  
fazni odziv

Zanju velja:

$$|H(-\Omega)| = |H(\Omega)|$$

$$\phi_H(-\Omega) = -\phi_H(\Omega)$$

recimo:  $x[n] = \cos(\Omega_1 n)$

sistem je podan z impulznim odzivom

$$\begin{aligned} y[n] &= h[n] * x[n] = \sum_{i=-\infty}^{\infty} h[i] \cdot \cos(\Omega_1(n-i)) = \frac{1}{2} \sum_{i=-\infty}^{\infty} h[i] (e^{j\Omega_1(n-i)} + e^{-j\Omega_1(n-i)}) = \\ &= \frac{1}{2} \left[ e^{j\Omega_1 n} \sum_{i=-\infty}^{\infty} h[i] e^{-j\Omega_1 i} + e^{-j\Omega_1 n} \sum_{i=-\infty}^{\infty} h[i] e^{j\Omega_1 i} \right] = \\ &= \frac{1}{2} \left[ \underbrace{e^{j\Omega_1 n} H(\Omega_1)}_{|H(\Omega_1)| e^{j\phi_H(\Omega_1)}} + \underbrace{e^{-j\Omega_1 n} H(-\Omega_1)}_{|H(\Omega_1)| e^{-j\phi_H(\Omega_1)}} \right] = |H(\Omega_1)| \frac{e^{j(\Omega_1 n + \phi_H(\Omega_1))} + e^{-j(\Omega_1 n + \phi_H(\Omega_1))}}{2} = \\ &= \underline{|H(\Omega_1)| \cdot \cos(\Omega_1 n + \phi_H(\Omega_1))} \leftarrow \text{sprememba amplitude in faze!} \end{aligned}$$

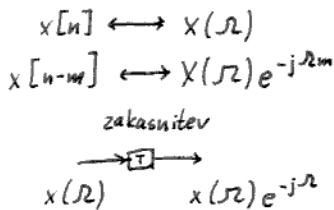
Določanje frekvenčnega odziva  $H(\Omega)$

a) iz impulznega odziva  $H(\Omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\Omega n}$

za kausalne:

$$H(\Omega) = \sum_{n=0}^{\infty} h[n] e^{-j\Omega n}$$

b) direktno iz diferenčne enačbe oz. sheme sistema



diferenčna enačba  $\xrightarrow{\text{TDFT}}$  algebrainska enačba

8.4.2013

c) iz sistemske funkcije z zamenjavo  $z \rightarrow e^{j\Omega}$

$$H(\Omega) = H(z) \Big|_{z=e^{j\Omega}}$$

5.5 z-transformacija

orodje za reševanje diferencialnih enačb

Dvostranska in enostranska z-transformacija

$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$   $z$  je kompleksna spremenljivka

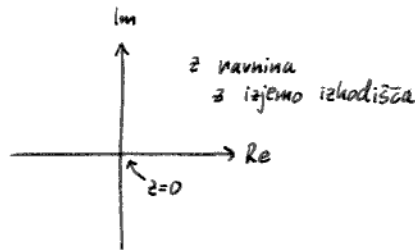
za desnostranske signale ( $x[n]=0$  za  $n < 0$ ):

$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$

Konvergenčno področje transformacije (ROC)

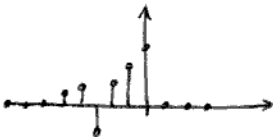
1. Časovno omejen signal

a) kavalni signal



$\frac{1}{z^n} \rightarrow z=0 \rightarrow \frac{1}{0}$

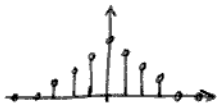
b) levostranski signal



konvergenca na celi ravnini razen ko  $|z| \rightarrow \infty$

$\frac{1}{z^n} \rightarrow \frac{1}{z^{-n}} \rightarrow z=\infty \rightarrow \frac{1}{0}$

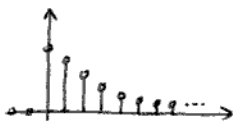
c) obojestranski signal



konvergenca na celi ravnini razen  $z=0$  in  $|z|=\infty$

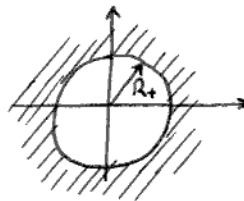
2. Časovno neomejen signal

a) kavalni signal

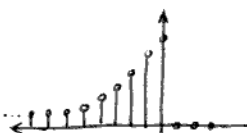


za konvergenco mora biti  $z$  dovolj velik

$|z| > R_+$

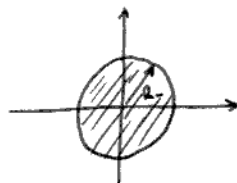


b) levostranski signal

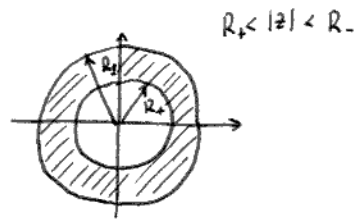
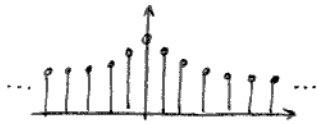


$z$  dovolj majhen

$|z| < R_-$



c) obojestranski signal



Zveza med  $X(z)$  in  $X(\Omega)$

$$z = r e^{j\Omega}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\Omega n}$$

$$X(z) \Big|_{z=e^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} = X(\Omega)$$

Lastnosti z transformacije

$$x_i[n] \xleftrightarrow{zT} X_i(z)$$

1. Linearnost

2. Premik  $x[n-n_0] \leftrightarrow X(z) \cdot z^{-n_0}$

3. Zasuk  $x[-n] \leftrightarrow X(z^{-1})$

4. Odvajanje po z  $n \cdot x[n] \leftrightarrow -z \frac{dX(z)}{dz}$

5. Konvolucija  $x_1[n] * x_2[n] \leftrightarrow X_1(z) \cdot X_2(z)$

6. množenje z  $a^n$   $a^n \cdot x[n] \leftrightarrow X\left(\frac{z}{a}\right)$

Nekaj značilnih signalov in transformi

1. Delta impulz  $\delta[n] \leftrightarrow 1$  za vse z

$$X_\delta(z) = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1$$

$$\delta[n-i] \leftrightarrow z^{-i} \quad \text{povsod, razen } z=0 \text{ za } i>0$$

$$\text{--- } |z|=\infty \text{ za } i<0$$

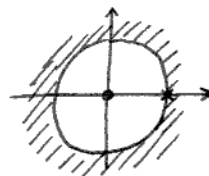
2. Enotina stopnica

$$u[n] - u[n-1] = \delta[n] \quad \text{za } |z| > 1$$

$$\uparrow zT$$

$$U(z) - U(z) z^{-1} = 1$$

$$U(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$



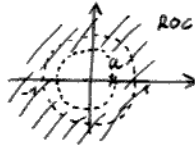
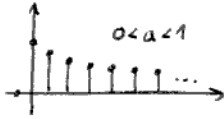
3. Desnostnanska eksponentna funkcija

$$a^n \cdot u[n] \leftrightarrow \frac{z}{z-a}$$

izračun:

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a \cdot z^{-1})^n = \frac{1-0}{1-a z^{-1}} \quad \text{če velja } |a z^{-1}| < 1$$

$$|z| > |a| \quad \text{ROC}$$

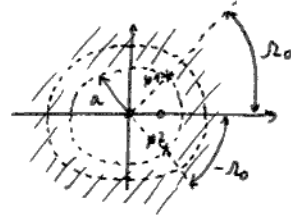


4. Dušeni kosinusni signal

$$x[n] = a^n \cdot \cos(\Omega_0 n) \cdot u[n]$$



$$\leftrightarrow \frac{z(z - a \cos \Omega_0)}{z^2 - 2a \cos \Omega_0 z + a^2}$$



5. Dušeni sinus

↑  
whatever

$$x[n] = a^n \cdot \sin(\Omega_0 n) \cdot u[n] \leftrightarrow \frac{z \cdot a \cdot \sin \Omega_0}{z^2 - 2a \cos \Omega_0 z + a^2}$$

ROC:  
 $|z| > |a|$

■ Zgled pravila o odvajanju

$$X(z) \leftrightarrow X[n]$$

$$x[n] = a^n \cdot u[n], \quad a < 1$$

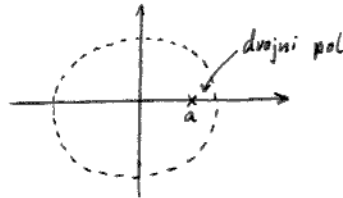
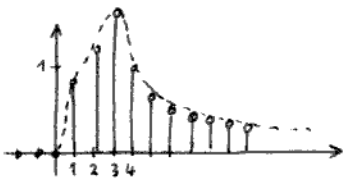
$$-z \frac{dX(z)}{dz} \leftrightarrow n \cdot x[n]$$

$$\downarrow zT$$

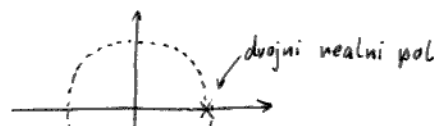
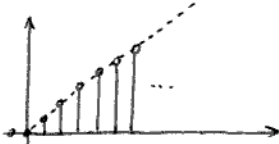
$$X(z) = \frac{z}{z-a}$$

$$-z \frac{dX(z)}{dz} = -z \frac{z-a - z}{(z-a)^2} = \frac{a \cdot z}{(z-a)^2} \leftrightarrow n \cdot a^n \cdot u[n] \quad \text{nežemo } a=0.8$$

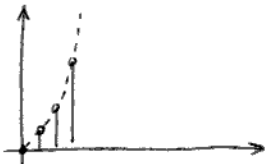
$a=0.8$



$a=1$



$a > 1$

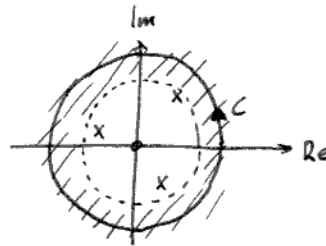


↑  
nekje tu pol ...  
x

Inverzna z-transformacija

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

C-skljenjena krivulja v ROC in obkroža izhodišče v pozitivnem smislu



$$x[n] = \sum_{i=1}^n \text{Res} [x(z) z^{n-1}]$$

$$\text{Res} [x(z) z^{n-1}] = \lim_{z \rightarrow p_i} \left[ \frac{1}{(m_i-1)!} \frac{d^{(m_i-1)}}{dz^{m_i-1}} \left( (z-p_i)^{m_i} x(z) z^{n-1} \right) \right]$$

za  $m_i = 1$

$m_i$  je red pola  $p_i$

$$\text{Res} [x(z) z^{n-1}] = \lim_{z \rightarrow p_i} [(z-p_i) x(z) z^{n-1}]$$

za  $m_i = 2$

$$\text{Res} [x(z) z^{n-1}] = \lim_{z \rightarrow p_i} \left[ \frac{d}{dz} \left( (z-p_i)^2 x(z) z^{n-1} \right) \right]$$

☒ Zgled:

$$X(z) = \frac{z}{z-a} \quad \text{ROC: } |z| > |a|$$

konvergenca zahteva, da velja  $n \geq 0$

$$x[n] = \text{Res} \left[ \frac{z}{z-a} \cdot z^{n-1} \right] = \lim_{z \rightarrow a} \left( (z-a) \frac{z}{z-a} z^{n-1} \right)$$

$$x[n] = a^n \quad \text{za } n \geq 0$$

$$\emptyset \quad \text{za } n < 0$$

☒ Zgled: uporaba tabele

$$X(z) = \frac{3z}{(z-0.6)(z+0.5)} \quad \text{ROC } |z| > 0.6$$

desnostranski signal!

$$X(z) = \frac{A}{z-0.6} + \frac{B}{z+0.5} = \frac{A \cdot z + 0.5A + Bz - 0.6B}{(z-0.6)(z+0.5)}$$

$$= \frac{18}{11} \frac{1}{z-0.6} + \frac{15}{11} \frac{1}{z+0.5}$$

$$\begin{aligned} A+B &= 3 \\ 0.5A - 0.6B &= 0 \end{aligned} \rightarrow \begin{aligned} B &= \frac{15}{11} \\ A &= \frac{18}{11} \end{aligned}$$

$$x[n] = \frac{18}{11} 0.6^{n-1} u[n-1] + \frac{15}{11} (-0.5)^{n-1} u[n-1]$$

## 5.6 Sistemaska funkcija $H(z)$

Definicija:

$$y[n] = x[n] * h[n]$$

↓ zT

$$Y(z) = X(z) \cdot H(z)$$

$$Y(z) = X(z) \cdot H(z)$$

•  $H(z)$  je z-transformirani impulznega odziva „nrtrega“ sistema

$$H(z) = \mathcal{Z}\{h[n]\}$$

$$H(z) = \frac{Y(z)}{X(z)}$$

### Določanje $H(z)$

1. Iz znanega impulznega odziva

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

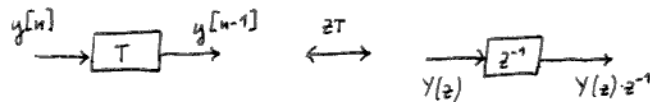
za kavzalne odzive

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

2. Iz blokovne sheme v z-prostoru

- seštevalniki in množilniki nespremenjeni

- zakasnitvi elementi se spremenijo v množilnike  $y[n-m] \leftrightarrow z^{-m} Y(z)$



3. S transformacijo diferencialne enačbe

$$y[n] = \sum_{i=0}^N b_i x[n-i] + \sum_{j=1}^M a_j y[n-j]$$

↓ zT

$$Y(z) = \sum_{i=0}^N b_i X(z) \cdot z^{-i} + \sum_{j=1}^M a_j Y(z) \cdot z^{-j}$$

$$\left. \begin{aligned} & \text{noti} \\ & \frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{i=0}^N b_i z^{-i}}{1 - \sum_{j=1}^M a_j z^{-j}} \end{aligned} \right\}$$

18.4.2013

### Poli in ničle sistemske funkcije

našo  $H(z)$  predelamo:  $H(z) \cdot \frac{z^N}{z^N} \cdot z^{M-N}$

$$H(z) = \frac{\sum_{i=0}^N b_i z^{N-i}}{z^M - \sum_{j=1}^M a_j z^{M-j}} \cdot z^{M-N} = \frac{b_0 z^N + \frac{b_1}{b_0} z^{N-1} + \frac{b_2}{b_0} z^{N-2} + \dots + \frac{b_N}{b_0}}{z^M - a_1 z^{M-1} - a_2 z^{M-2} - \dots - a_M} \cdot z^{M-N}$$

ničle polinoma v števcu so ničle sistemske fje.

poli sistemske fje. so ničle polinoma v imenovalcu

$$H(z) = b \frac{(z-p_1)(z-p_2)(z-p_3)\dots(z-p_N)}{(z-p_1)(z-p_2)(z-p_3)\dots(z-p_M)} z^{M-N}$$

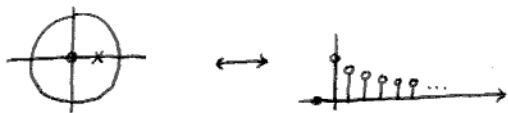
Vpliv lege ničel in polov  $H(z)$  na lastnosti sistema

1. Stabilnost

sistem je stabilen, če so vsi poli znotraj enotske krožnice

$|p_i| < 1$  za vse  $i \iff$  STABILNOST

$H(z) = \sum_{j=1}^M \frac{A_j z}{z - p_j} \iff h[n] = \sum_{j=1}^M A_j p_j^n u[n]$

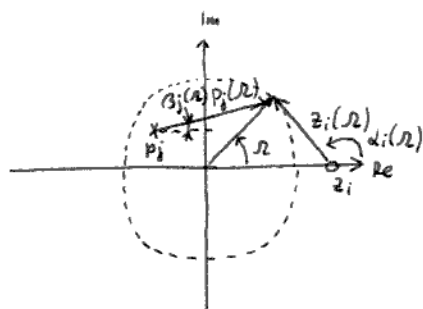


ničle ne vplivajo na stabilnost, ležijo lahko kjenkoli

2. Poli in ničle do konstante  $b_0$  določajo frekvenčni odziv  $H(\omega)$

$H(z) = \frac{\prod_{i=1}^N (z - z_i)}{\prod_{j=1}^M (z - p_j)} b_0 z^{M-N}$

$H(\omega) = H(z)|_{z=e^{j\omega}} = b_0 \frac{\prod_{i=1}^N (e^{j\omega} - z_i)}{\prod_{j=1}^M (e^{j\omega} - p_j)} e^{j\omega(M-N)}$



$e^{j\omega} - z_i = z_i(\omega) = |z_i(\omega)| e^{j\alpha_i(\omega)}$

$e^{j\omega} - p_j = |p_j(\omega)| e^{j\beta_j(\omega)}$

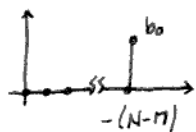
$H(\omega) = b_0 \underbrace{\frac{\prod_{i=1}^N |z_i(\omega)| e^{j\alpha_i(\omega)}}{\prod_{j=1}^M |p_j(\omega)| e^{j\beta_j(\omega)}}}_{\text{amplituda}} e^{j\omega(M-N)} = |H(\omega)| e^{j\phi_H(\omega)}$

poli in ničle v izhodišču ne vplivajo na amplitudni potek, samo na fazo (linearna)

$\phi_H(\omega) = \sum_{i=1}^N \alpha_i(\omega) - \sum_{j=1}^M \beta_j(\omega) + \omega(M-N) \pm k\pi$

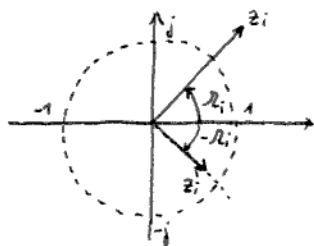
↑  
odvisen od  $b_0$   
1 ali 0

$M-N$  določa o začetku  $h[n]$



$H(z) = \frac{b_0 z^N}{z^M} z^{M-N} = (b_0 z^{N-M} + (b_1 + b_0 a_1) z^{N-M-1}) z^{M-N}$

3. Preslikava ("zrcaljenje") ničel



$z_i^* = \frac{1}{z_i} = \frac{1}{|z_i| e^{j\alpha_i}} = \frac{1}{|z_i|} e^{-j\alpha_i}$

preslikava ničel preko enotske krožnice ne vpliva na obliko  $|H(\omega)|$  ampak spremeni konstanto in fazni potek za dokaz obrni stran.....

$$H(z) = H_1(z) \cdot H_2(z) \rightarrow H_2(z) = (z - z_i)(z - z_i^*)$$

$$H'(z) = H_1(z) \cdot H_2'(z) \rightarrow H_2'(z) = (z - \frac{1}{z_i})(z - \frac{1}{z_i^*})$$

$$H_2'(z) = \frac{z}{z_i} (z_i - \frac{1}{z}) (\frac{z_i^* - 1}{z_i^*}) \frac{z}{z_i^*} = \frac{z^2}{|z_i|^2} (z^{-1} - z_i)(z^{-1} - z_i^*)$$

$$H_2'(\Omega) = H_2'(z) \Big|_{z=e^{j\Omega}} = \frac{e^{j2\Omega}}{|z_i|^2} (e^{-j\Omega} - z_i)(e^{-j\Omega} - z_i^*) = \frac{e^{j2\Omega}}{|z_i|^2} H_2(-\Omega) =$$

$$|H_2'(\Omega)| = \frac{1}{|z_i|^2} |H_2(-\Omega)| =$$

$$\underline{\underline{\phi_2'(\Omega) = 2\Omega - \phi_2(\Omega)}}$$

$$\underline{\underline{|H_2'(\Omega)| = \frac{1}{|z_i|^2} |H(\Omega)|}}$$

za realno ničlo:

$$|H_2'(\Omega)| = \frac{1}{|2a|}$$

$$\phi_2'(\Omega) = \pi + \Omega - d_i(\Omega)$$

Sistemi z minimalno fazo imajo ničle znotraj enotske krožnice

-||- z maksimalno fazo -||- zunaj enotske krožnice

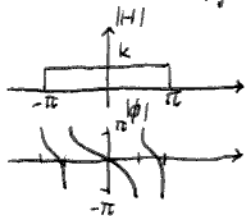
#### 4. Fazni sukabnik (all-pass filter)

$$|H(\Omega)| = \text{konstanta} = 1 \text{ (ponavadi)} \quad -\pi < \Omega < \pi$$

1. vsi poli znotraj enotske krožnice  $|p_j| < 1 \quad j=1 \dots M$

2. vsak pol uparjen z ničlo zunaj krožnice

$$z_j = \frac{1}{p_j^*}; \quad j=1 \dots M$$



$$\text{za } |H(\Omega)| = 1$$

$$|b_0| = \prod_{j=1}^M |p_j|$$

$$1) \quad H(z) = \frac{\prod_{j=1}^M (z - \frac{1}{p_j^*})(z - \frac{1}{p_j})}{\prod_{j=1}^M (z - p_j)(z - p_j^*)}$$

$$H_j(\Omega) = \frac{(e^{j\Omega} - \frac{1}{p_j^*})(e^{j\Omega} - \frac{1}{p_j})}{(e^{j\Omega} - p_j)(e^{j\Omega} - p_j^*)} = \frac{e^{j\Omega}(p_j - e^{-j\Omega}) e^{j\Omega}(p_j^* - e^{-j\Omega})}{p_j(e^{j\Omega} - p_j) p_j^*(e^{j\Omega} - p_j^*)} =$$

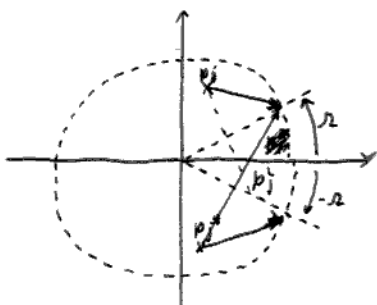
$$2) \quad e^{j\Omega} - p_j = |p_j(\Omega)| e^{j\beta_j(\Omega)}$$

$$e^{j\Omega} - p_j^* = |p_j(\Omega)| e^{-j\beta_j(\Omega)}$$

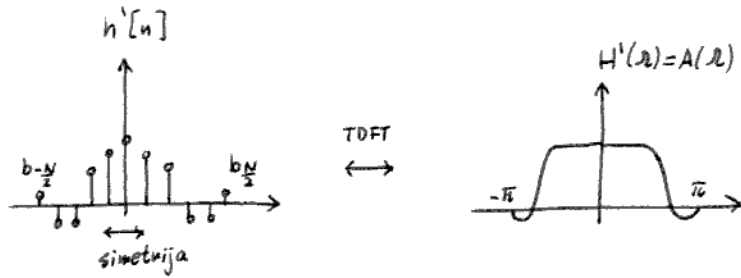
$$3) \quad H_j(\Omega) = \frac{e^{j2\Omega} |p_j| e^{-j\beta_j(\Omega)} |p_j| e^{j\beta_j(\Omega)}}{|p_j|^2 |p_j| e^{j\beta_j(\Omega)} |p_j| e^{-j\beta_j(\Omega)}}$$

$$H_j(\Omega) = \frac{e^{j2\Omega} (e^{-j\Omega} - p_j^*) (e^{-j\Omega} - p_j)}{|p_j|^2 (e^{j\Omega} - p_j) (e^{j\Omega} - p_j^*)}$$

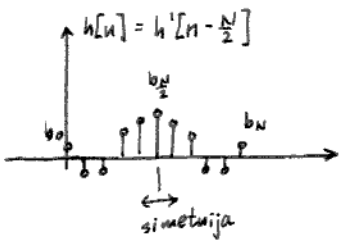
$$= \frac{1}{|p_j|^2} e^{j[2\Omega - 2\beta_j(\Omega) - 2\beta_j'(\Omega)]}$$



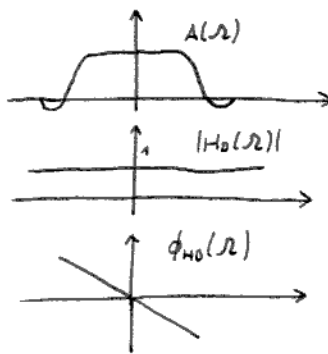
5. Sistemi z Linearno fazo



ta odziv je seveda nekavsalen, Lahko ga pa zakasnimo



$$H(\omega) = A(\omega) e^{-j\omega \frac{N}{2}} \quad H_0(\omega) = e^{-j\omega \frac{N}{2}}$$



simetričen h[n], FIR!

Tip 1 simetrični koeficienti  $b_i = b_{N-i}$   
 $N$  je sod - centralni impulz  $b_{\frac{N}{2}}$

22.4.2013

$$H(\omega) = \sum_{n=0}^N b_n e^{-j\omega n}$$

$$H(\omega) = \sum_{n=0}^{\frac{N}{2}-1} b_n e^{-j\omega n} + b_{\frac{N}{2}} e^{-j\omega \frac{N}{2}} + \sum_{n=\frac{N}{2}+1}^N b_n e^{-j\omega n}$$

$$+ \sum_{n=\frac{N}{2}+1}^N b_n e^{-j\omega n} =$$

$$= e^{-j\omega \frac{N}{2}} \left[ b_{\frac{N}{2}} + \sum_{n=0}^{\frac{N}{2}-1} b_n e^{-j(\frac{N}{2}-n)\omega} + \sum_{n=\frac{N}{2}+1}^N b_n e^{-j(\frac{N}{2}-n)\omega} \right] = e^{-j\omega \frac{N}{2}} A(\omega)$$

- Upeljemo substitucijo  $\frac{N}{2} - n = k$   $k(n=0) = \frac{N}{2}$
- za prvo vsoto:  $n = \frac{N}{2} - k$   $k(n = \frac{N}{2} - 1) = 1$
  - za drugo vsoto:  $\frac{N}{2} - n = -k$   $k(\frac{N}{2} + 1) = 1$   
 $n = \frac{N}{2} + k$   $k(N) = \frac{N}{2}$

$$A(\omega) = b_{\frac{N}{2}} + \sum_{k=\frac{N}{2}}^1 b_{\frac{N}{2}-k} e^{jk\omega} + \sum_{k=1}^{\frac{N}{2}} b_{\frac{N}{2}+k} e^{-jk\omega} =$$

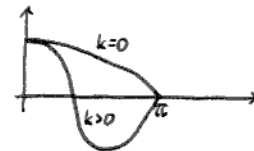
$$= b_{\frac{N}{2}} + \sum_{k=1}^{\frac{N}{2}} b_{\frac{N}{2}-k} (e^{jk\omega} + e^{-jk\omega}) = b_{\frac{N}{2}} + 2 \sum_{k=1}^{\frac{N}{2}} b_{\frac{N}{2}-k} \cos(k\omega)$$

Tip 2 simetrični koeficienti  $b_i = b_{N-i}$   
 $N$  je lih,  $n$  centralnega impulza

$$H(\Omega) = \sum_{n=0}^N b_n e^{-j\Omega n} = e^{-j\frac{N}{2}\Omega} \sum_{n=0}^{\frac{N-1}{2}} b_n e^{j(\frac{N}{2}-n)\Omega} + e^{-j\frac{N}{2}\Omega} \sum_{n=\frac{N+1}{2}}^N b_n e^{j(\frac{N}{2}-n)\Omega}$$

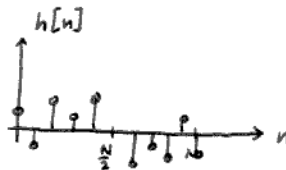
$$A(\Omega) = \sum_{k=0}^{\frac{N-1}{2}} b_{\frac{N-1}{2}-k} e^{j(\frac{1}{2}+k)\Omega} + \sum_{k=0}^{\frac{N-1}{2}} b_{\frac{N-1}{2}+k} e^{-j(\frac{1}{2}+k)\Omega}$$

$$A(\Omega) = 2 \sum_{k=0}^{\frac{N-1}{2}} b_{\frac{N-1}{2}-k} \cos\left(\left(\frac{1}{2}+k\right)\Omega\right)$$

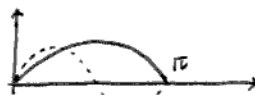


vedno ničla pri  $\pi$

Tip 3 anti-simetrični koeficienti  $b_i = -b_{N-i}$   
 $N$  je sod



$$A(\Omega) = 2 \sum_{k=1}^{\frac{N}{2}} b_{\frac{N}{2}-k} \sin(k\Omega)$$



ničla pri 0 in  $\pi$

Bandpass filter!

Tip 4 anti-simetrični  
 $N$  je lih

$$A(\Omega) = 2 \sum_{k=0}^{\frac{N-1}{2}} b_{\frac{N-1}{2}-k} \sin\left(\left(\frac{1}{2}+k\right)\Omega\right)$$

ničla pri 0

$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_2 z^{-(N-2)} + b_1 z^{-(N-1)} + b_0 z^{-N}$  ničle sistemske funkcije

$$b_0 + b_1 \left(\frac{1}{z}\right) + b_2 \left(\frac{1}{z}\right)^2 + \dots + b_2 \left(\frac{1}{z}\right)^{N-2} + b_1 \left(\frac{1}{z}\right)^{N-1} + b_0 \left(\frac{1}{z}\right)^N = 0 \quad | \cdot z^N$$

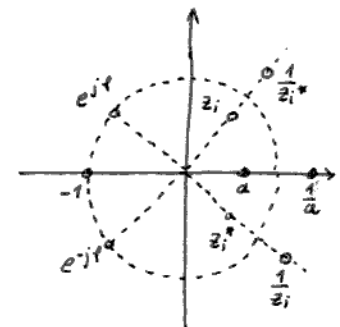
$$b_0 z^N + b_1 z^{N-1} + b_2 z^{N-2} + \dots + b_2 z^2 + b_1 z + b_0 = 0$$

ničle nastopajo recipročno  $\rightarrow z_i$  je ničla, tudi  $\frac{1}{z_i}$  je ničla  $H(z)$

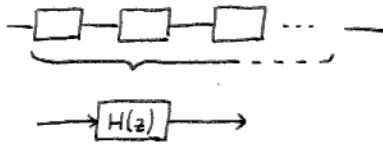
- vsi poli v izhodišču (FIR-odziv)
- ničle nastopajo v recipročnih vrednostih (parah), razen pri  $\pm 1$

Možne lege ničel:

- par na realni osi:  $z=a$  in  $z=\frac{1}{a}$
- kompleksno konjugiran par na enotski krožnici  $z=e^{j\theta}$ ,  $z=e^{-j\theta}$
- štiri kompleksne ničle:  $|z|>1$   $z_i, z_i^*, \frac{1}{z_i}$  in  $\frac{1}{z_i^*}$



6. Kaskadna vezava sistemov



$$H(z) = H_1(z) \cdot H_2(z) \cdot \dots \cdot H_k(z)$$

$\delta$  ničle in poli se dodajajo med stopnjami ...

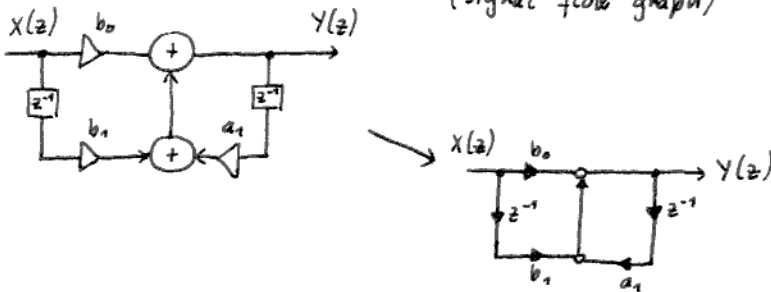
7. Common Sense!

$M \geq N!$

6. Diskretni filtri

6.1 Blokovna shema in diagram poteka signala

(signal flow graph)



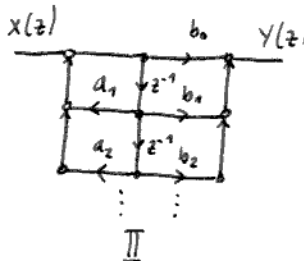
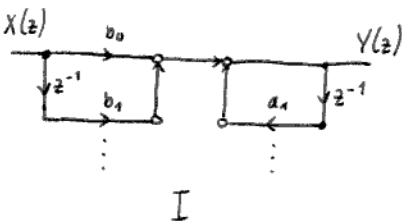
Transpozicija diagrama pretoka

- obrnemo smer pretoka v vseh vejah
- razvejišča se spremenijo v sestevalnike in obratno
- vhod in izhod se zamenjata

včasih se zgodi, da je transponirana shema boljša, bolj stabilna, ...

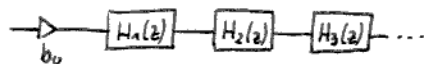
6.2 Strukture filtrov

Direktna struktura I in II



Kaskadna vezava

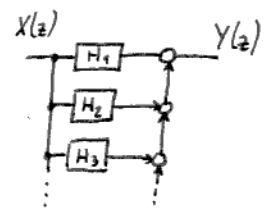
za IIR filtre



$$H(z) = b_0 \prod_{i=1}^K H_i(z)$$

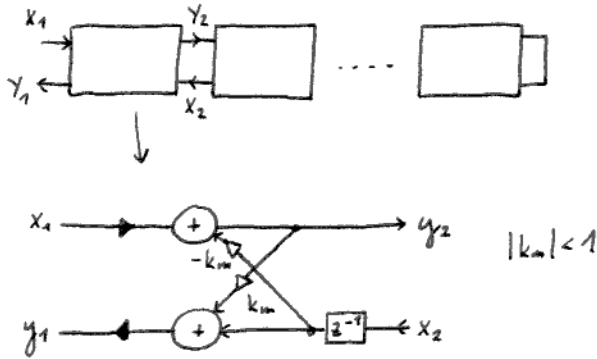
Vzporedna vezava

$$H(z) = \sum_{i=1}^L \frac{b_{0i} + b_{1i} z^{-1}}{1 - a_{1i} z^{-1} + a_{2i} z^{-2}}$$

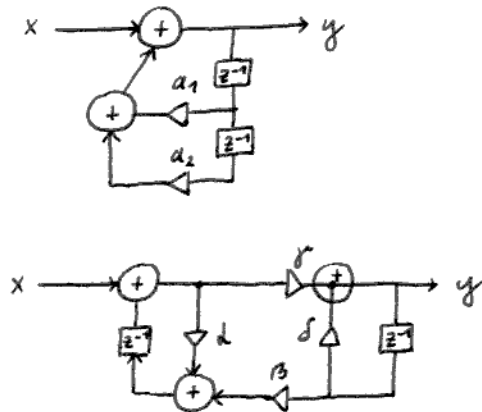


Druge strukture

Lestvičasti filter - lattice filter realizacija FIR in IIR filter



Sklopljena struktura 2. reda (coupled form)



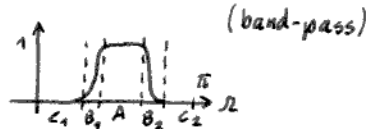
Značilni predstavniki diskretnih filtrov

• nizko sito



A - prepustni pas  
B - prehodni pas  
C - zaporni pas

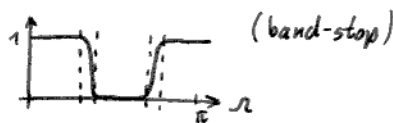
• pasovno prepustno sito



• visoko sito



• pasovno zaporno sito



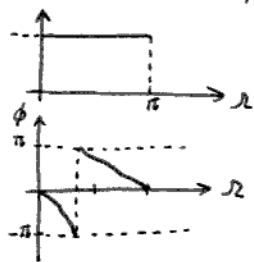
• zarežno sito

(notch-filter)

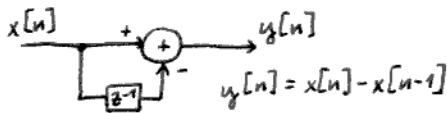


• fazni - skakalnik

(all-pass)



• diferenciator



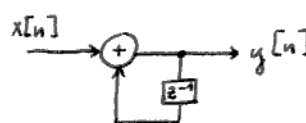
$H(z) = \frac{z-1}{z}$      $H(\omega) = 2j \sin \frac{\omega}{2} \cdot e^{-j \frac{\omega}{2}}$

zvezni diferenciator

$y(t) = \frac{dx(t)}{dt}$

$H(\omega) = j\omega$

• diskretni sumator (akumulator)



$y[n] = x[n] + y[n-1] = \dots$

$y[n] = \sum_{i=-\infty}^n x[i]$

$H(z) = \frac{z}{z-1}$

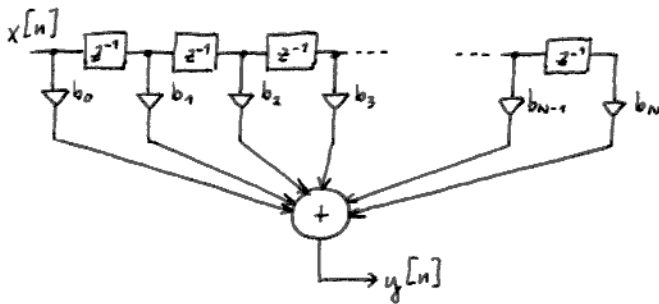
$H(\omega) = \frac{1}{2j \sin \frac{\omega}{2}} \cdot e^{j \frac{\omega}{2}}$

## Načrtovanje FIR filtrov

- imajo končen odziv, so vedno stabilni
- dolžina odziva je  $N+1$  in je enaka številu koeficientov
- poli ležijo v izhodišču
- lahko imajo linearno fazo
- primerni so za adaptivne izvedbe

### Strukture

a) transverzalni filter (kanonična oblika)

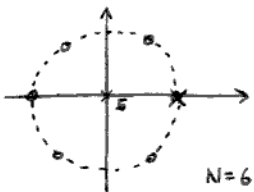


$$H(z) = \sum_{i=0}^N b_i z^{-i} = \frac{\sum_{i=0}^N b_i z^{N-i}}{z^N}$$

b) nekonzivna izvedba

- filter za tekoče povprečje

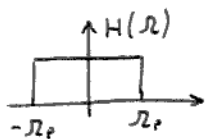
$$H(z) = \frac{1}{N} \sum_{i=0}^{N-1} z^{-i} = \frac{1}{N} (1 + z^{-1} + z^{-2} + \dots + z^{-(N-1)}) = \frac{1}{N} \frac{1 - z^{-N}}{1 - z^{-1}} = \frac{1}{N} \frac{z^N - 1}{z^{N-1}(z - 1)}$$



## Načrtovanje FIR filtrov z oknjenjem

Idealno nizko sito:

$$H(\omega) = \begin{cases} 1 & \text{za } |\omega| < \omega_p \\ 0 & \text{sicer} \end{cases}$$



$$h[n] = \frac{\sin(\omega_p n)}{\pi n}$$

$$h[0] = \frac{\omega_p}{\pi}$$

odziv je simetričen in neskončen, zato ga omejimo in pomaknemo v desno, da ga lahko realiziramo:

$$h[n] = \begin{cases} \frac{\sin(\omega_p (n - \frac{N}{2}))}{\pi (n - \frac{N}{2})} & \text{za } 0 \leq n \leq N \\ 0 & \text{sicer} \end{cases}$$

nezanje povzroči neke visoke harmonike (Gibbs-ov efekt)

rezanje predstavimo z okuženjem

$$h[n] = h_i[n] \cdot w[n]$$

↑ TDFT

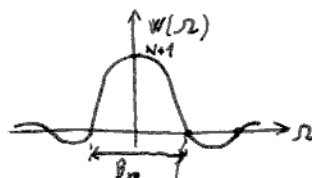
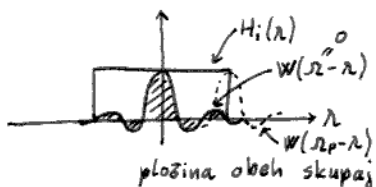
$$H(\Omega) = \frac{1}{2\pi} H_i(\Omega) * W(\Omega) = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} H_i(\Omega - \lambda) W(\lambda) d\lambda$$

$$W(\Omega) = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} e^{-j\Omega n} = \frac{\sin(\frac{N+1}{2} \cdot \Omega)}{\sin \frac{\Omega}{2}}$$

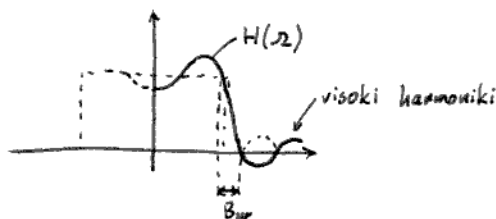
pravokotno okno

13.5.2013

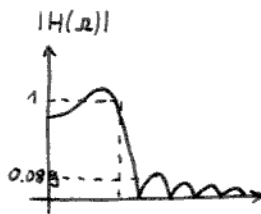
konvolucija:



nižla  
 $\pi = \frac{N+1}{2} \Omega$   
 $\Omega = \frac{2\pi}{N+1}$   
 $B_w = \frac{4\pi}{N+1}$



abs  
dB



dušenje  $20 \cdot \log(0.089) \approx -21$  dB (neodvisno od širine okna (N))

mного bolje se obnesejo okenske funkcije

Kosinusna okna

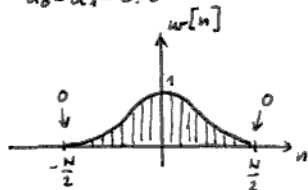
$$w[n] = \begin{cases} a_0 + a_1 \cos(\frac{2\pi}{N} \cdot n) & ; \text{za } |n| \leq \frac{N}{2} \\ 0 & ; \text{sicer} \end{cases}$$

• von Hannovo okno

$$a_0 = a_1 = 0.5$$

$$B_w = \frac{8\pi}{N}$$

dušenje stranskega snopa 43.9 dB



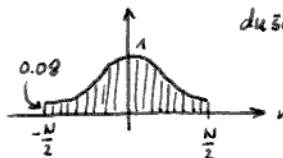
• Hammingovo okno

$$a_0 = 0.54$$

$$a_1 = 0.46$$

$$B_w = \frac{8\pi}{N}$$

dušenje 54.5 dB



• Blackmanovo okno

$$0.42 + 0.5 \cos(\frac{2\pi}{N} \cdot n) + 0.08 \cos(\frac{4\pi}{N} \cdot n)$$

$$B_w = \frac{5.56 \pi}{N}$$

dušenje 75.3 dB

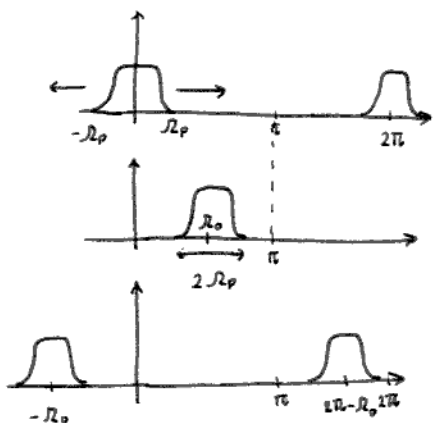
• Kaiserjevo, Gaussovo, Žebisevo... okno

Bistveni pravili pri načrtovanju z okenškimi fjami

1. izbina okna vpliva na minimalno slabljenje v zaponi  
 X večje slabljenje razširi prehodno območje
2. dolžina okna oziroma centralni snop  $W(\Omega)$   
 z dolžino vplivamo na širino prehodnega področja



b) Band-Pass sito

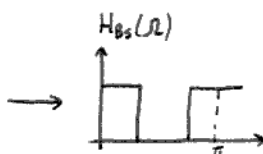
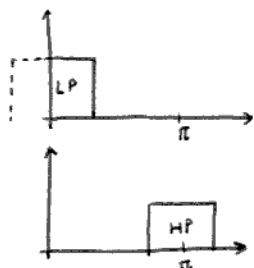
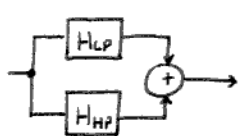


$$h[n] \cdot e^{j\Omega_0 n}$$

$$h_{BP}[n] = h[n] (e^{j\Omega_0 n} + e^{-j\Omega_0 n}) = h[n] \cdot 2 \cos(\Omega_0 n)$$

$$h[n] \cdot e^{-j\Omega_0 n}$$

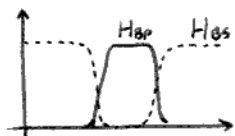
c) Band-Stop sito



$$H(z) = H_{LP}(z) + H_{HP}(z) =$$

$$= \sum_{i=0}^N b_{LP,i} z^{-i} + \sum_{i=0}^N b_{HP,i} z^{-i} = \sum_{i=0}^N (b_{LP,i} + b_{HP,i}) z^{-i}$$

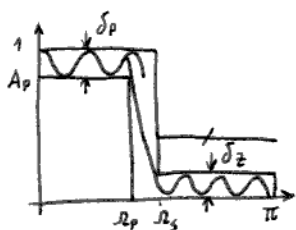
Za sodi nedN lahko uporabimo odštevanje:



$$H \cdot e^{-j\frac{N}{2}\Omega}$$

$$H_{BS} = e^{-j\frac{N}{2}\Omega} A_{BP}(\Omega)$$

Načrtovanje FIR sit z enakomerno valovitostjo



Postopek Parks-McLlallen

$N, \Omega_p, \Omega_s$ , nazmerje  $\delta_p/\delta_s$

$$A_p [dB] : \delta_p = 1 - 10^{\frac{A_p}{20}}$$

$$A_s [dB] : \delta_s = 10^{\frac{A_s}{20}}$$

6.5. Načrtovanje IIR filtrovLastnosti:

- imajo rekurzivno strukturo
- so potencialno nestabilni
- odziv je neskončen vendar imajo krajše zakasnitve
- niži ned za doseganje zahtevane karakteristike
- večja občutljivost na kvantizacijo
- ne moremo doseči linearne faze
- lahko načrtujemo fazni sukelnik

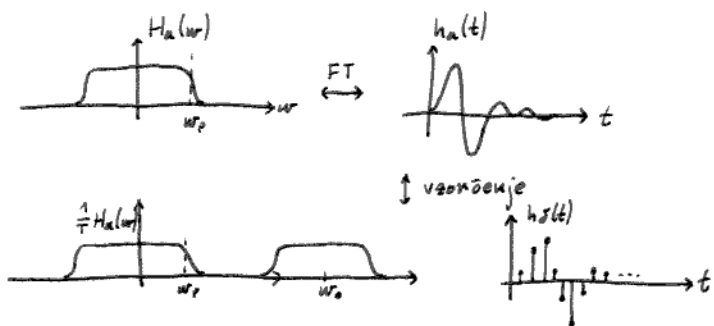
Strukture:

- Direktna struktura I in II
- Kaskadna vezava filtrov 2. reda (tudi 1. reda)
- Lestvičasta (rešetkasta) struktura
- Paralelna struktura (uporabljamo zelo redko)

Impulzno invariantna metoda:

$h_x(t)$  imajo odziv analognega prototipa

$$h[n] = h_a(nT)$$



$$H(\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a(\omega - k\omega_0) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(\frac{\Omega - k2\pi}{T}\right)$$

uporabna za filtre, katerih odziv monotono upada! (nizka sira, ki imajo le pole)

- Butterworthov, Čebišev I  
(brez valovitosti v zaponi!)

$$H(z) = \sum_{n=0}^{\infty} h_a(nT) z^{-n}$$

↑  
tega nimamo, ponavadi imamo  $H_a(s)$

v razcepljeni obliki:

$$H_a(s) = \sum_{i=1}^N \frac{c_i}{s - s_{pi}} = b_0 \prod_{i=1}^N \frac{1}{(s - s_{pi})}$$

$$c_i = \text{Res } H_a(s)_{s=s_{pi}}$$

$$h_a(t) = u(t) \cdot \sum_{i=1}^N c_i e^{s_{pi} \cdot t} \rightarrow h[n] = h_a(nT) = u[n] \sum_{i=1}^N c_i e^{s_{pi} \cdot Tn} \xrightarrow{zT} H(z) = \sum_{i=1}^N c_i \mathcal{Z}\{u[n] e^{s_{pi} \cdot Tn}\}$$

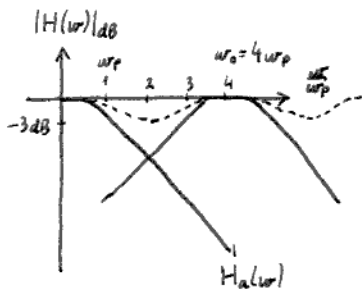
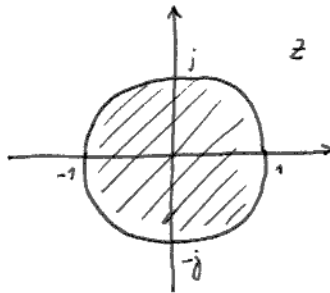
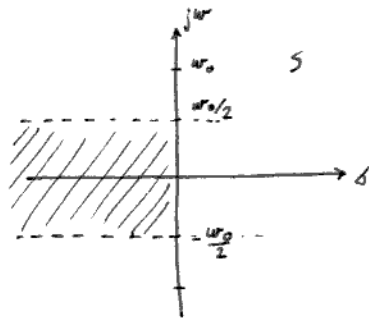
20.5.2013

$$u[n] \cdot e^{s_{pi} T n} \leftrightarrow \frac{1}{1 - e^{s_{pi} T} z^{-1}}$$

končno:

$$H(z) = \sum_{i=1}^N \frac{c_i}{1 - (e^{s_{pi} T} z^{-1})}$$

$$\underline{z_{pi} = e^{s_{pi} T}}$$

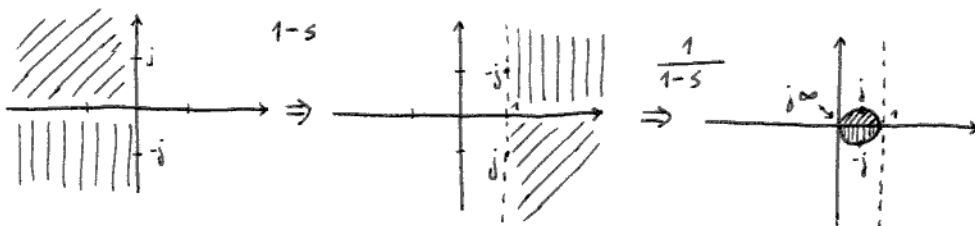


samo za monotono upadajoče filtre

Bilinearna transformacija:

$$z = \frac{1+s}{1-s} \quad s \text{ je normaliran } \frac{\Delta}{\omega_n} \leftarrow \text{si izberemo (buje frekvenca pola)}$$

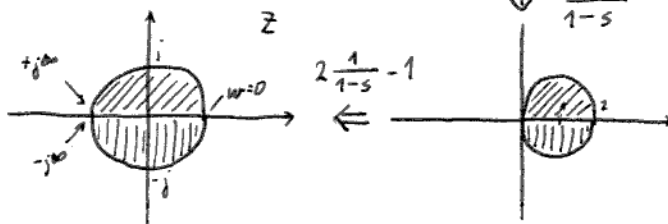
$$z = 2 \frac{1}{1-s} - 1$$



obratna transformacija:

$$s = \frac{z-1}{z+1} = \frac{\Delta}{\omega_n}$$

$$s = \omega_n \frac{z-1}{z+1}$$



$$H(z) = H_a\left(\omega_n \frac{z-1}{z+1}\right)$$

naje neposredno določimo pole in nize

poli:

$$z_{pi} = \frac{1 + \frac{s_{pi}}{\omega_n}}{1 - \frac{s_{pi}}{\omega_n}}$$

nize:

$$z_{ni} = \frac{1 + \frac{s_{ni}}{\omega_n}}{1 - \frac{s_{ni}}{\omega_n}}$$

$$H_a(s) = b_0 \frac{\prod_{i=1}^N (s - s_{ni})}{\prod_{j=1}^M (s - s_{pj})} \quad N \leq M$$

$$H(z) = b_0 (1+z^{-1})^{M-N} \frac{\prod_{i=1}^N (1 - z_{ni} z^{-1})}{\prod_{j=1}^M (1 - z_{pj} z^{-1})}$$

$s=0 \rightarrow z=1$

$b_0$  določimo iz enakosti

$$H(1) = H_a(0)$$

$z \quad s$

Stiskanje frekvenc:

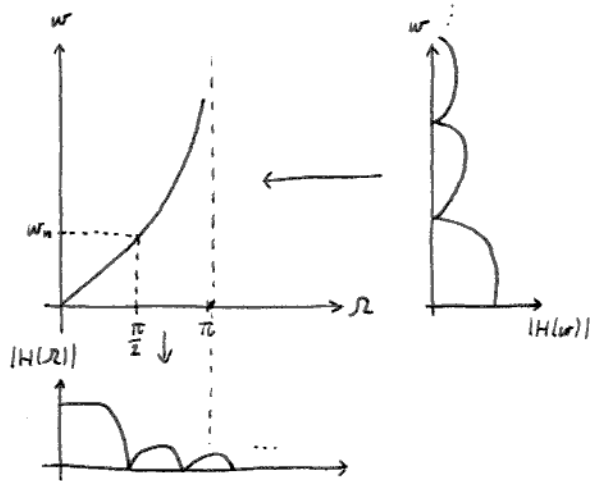
$\omega \in (0, \infty) \rightarrow \Omega \in (0, \pi)$

$$s = w_n \frac{z-1}{z+1} \quad ; \quad s = j\omega$$

$$z = e^{j\Omega}$$

$$j\omega = w_n \frac{e^{j\Omega} - 1}{e^{j\Omega} + 1} = \dots =$$

$$\omega = w_n \tan \frac{\Omega}{2} \quad \Omega = 2 \arctan \frac{\omega}{w_n}$$



Postopek:

1. Na osnovi  $\Omega_p, \Omega_s, A_p, A_s$  določimo analogne  $\omega_p, \omega_s$  (PASS/STOP BAND)

$$\omega_p = w_n \tan \frac{\Omega_p}{2}$$

$$\omega_s = w_n \tan \frac{\Omega_s}{2}$$

2. Na osnovi  $\omega_p, \omega_s, A_p$  in  $A_s$  določimo vrsto in nred analognega filtra

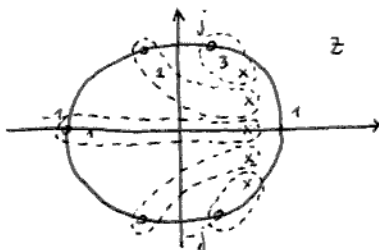
3. z bilinearno preslikavo določimo pole in ničle v z-ravnini

$$z_{pi} = \frac{1 + \frac{s_{pi}}{w_n}}{1 - \frac{s_{pi}}{w_n}} \quad \text{enako za ničle}$$

4. Zapišemo  $H(z)$

5. Izračunamo  $b_0$ , da ustreza danim zahtevam

6. Izberemo ustrezno strukturo za realizacijo (običajno kaskada celic drugega reda)



$$H_i(z) = \frac{1 + b_{1i} z^{-1} + b_{2i} z^{-2}}{1 - a_{1i} z^{-1} - a_{2i} z^{-2}}$$

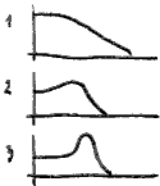
$$(1 - n_i z^{-1})(1 - n_i^* z^{-1}) = 1 - 2|n_i| \cos \Omega_{ni} z^{-1} + |n_i|^2 z^{-2}$$

$n_i, n_i^*$  in  $p_i, p_i^* \quad | \quad n_i = |n_i| e^{j\Omega_{ni}}$   
 $p_i = |p_i| e^{j\Omega_{pi}}$

$$n_i^* = |n_i| e^{-j\Omega_{ni}}$$

$$p_i^* = |p_i| e^{-j\Omega_{pi}}$$

$$\begin{bmatrix} b_{1i} = -2|n_i| \cos \Omega_{ni} \\ b_{2i} = |n_i|^2 \\ a_{1i} = 2|p_i| \cos \Omega_{pi} \\ a_{2i} = |p_i|^2 \end{bmatrix}$$



## 7. Sistemi z različnimi vzorčevalnimi

### frekvencami

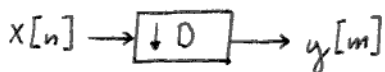
(multi-rate systems)

### 7.1. Razlogi za spremembo vzor. frekvence

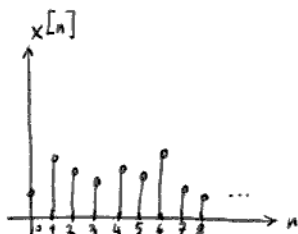
- prilagajanje med različnimi sistemi  
 telefonski govor 8 kHz (300 ÷ 3400 Hz)  
 audio signal 32 kHz - za radio  
 44,1 kHz - CD  
 48 kHz - DAT
- povečanje učinkovitosti obdelave  
 prilagajanje pasovni širini signala
- zmanjšanje zahtevnosti sit proti prekrivanju pri nekonstrukciji in zajemanju

### 7.2. Decimacija

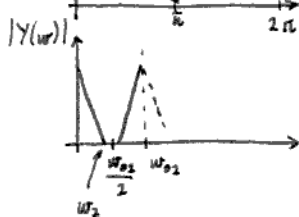
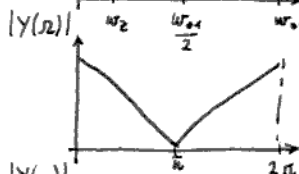
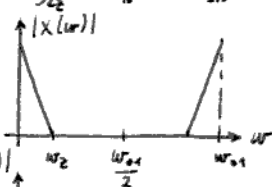
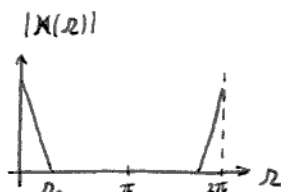
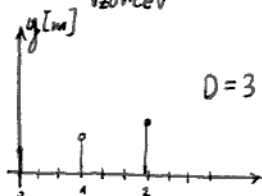
na D vzorcev spustimo D-1 originalnih



$$f_{o2} = \frac{f_{o1}}{D} \geq 2f_z \leftarrow \text{zgornja frekvenca signala}$$



spustimo nekaj vzorcev



če bi spustili še več vzorcev (recimo D=4) bi prišlo do prekrivanja v spektru

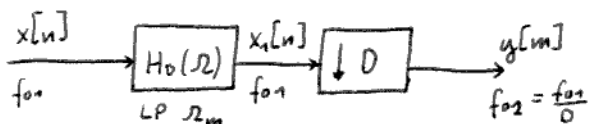
pogoj:

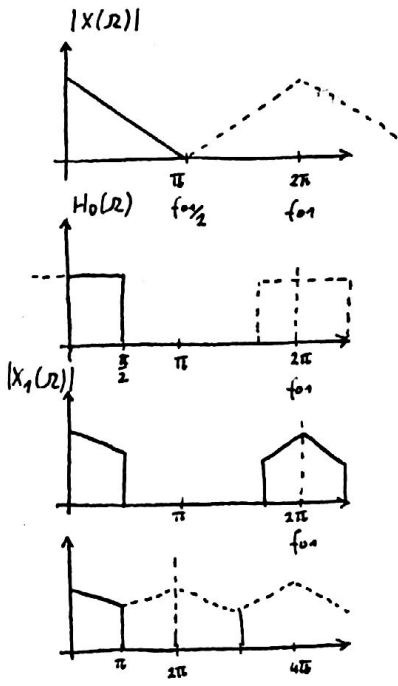
$$\Omega_z \leq \pi$$

$$\omega_z \leq \pi f_{o2}$$

spekter signala  $x[n]$  moramo pred decimacijo omejiti

$$\Omega_m = \frac{\omega_z}{f_{o1}} \leq \pi \frac{f_{o2}}{f_{o1}} = \frac{\pi}{D}$$



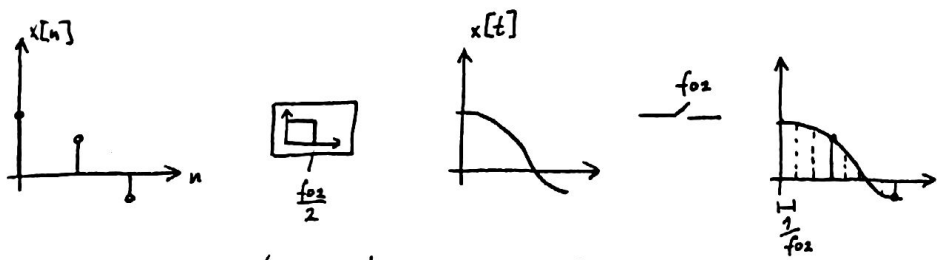


27.5.2013

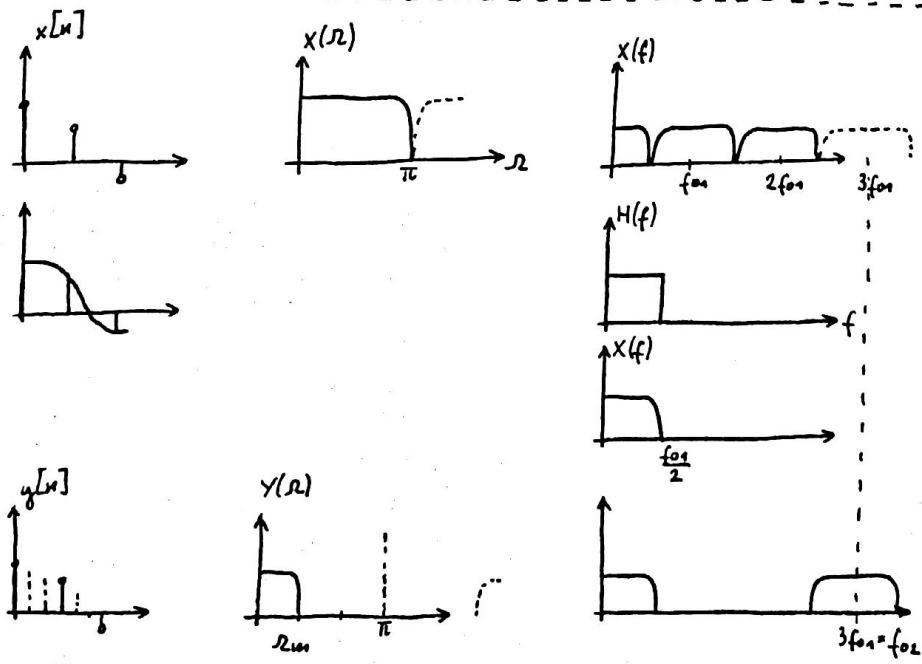
7.3. Interpolacija

I-1 novih vzorcev

$$I = \frac{f_{o2}}{f_{o1}}$$

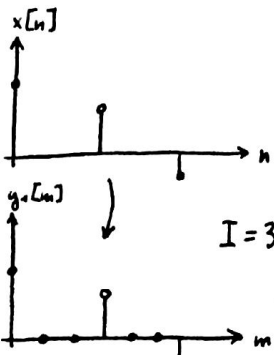


to pozetje je nesmiselno ...



$$\Omega_m = \frac{\omega_s}{f_{o2}} = \frac{2\pi f_{o1}}{f_{o2} \cdot 2} = \pi \cdot \frac{f_{o1}}{f_{o2}}$$

To lahko izvedemo tudi digitalno:

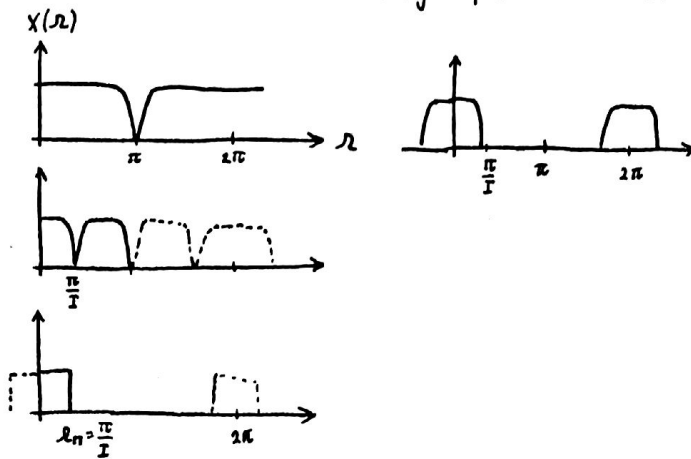


$I=3$  faktor interpolacije

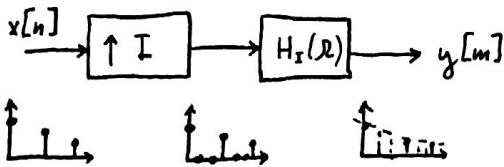
$$y_1[m] = \begin{cases} x[n] & \text{za } m=n \cdot I \\ 0 & \text{za } m \neq n \cdot I \end{cases}$$

spekter  $Y_1(\Omega) = \sum_{m=-\infty}^{\infty} y_1[m] \cdot e^{-j\Omega m} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n I} = \dots$

$Y_1(\Omega) = X(I \cdot \Omega)$  ← skrajnje frekvenčne osi



predstavitel:



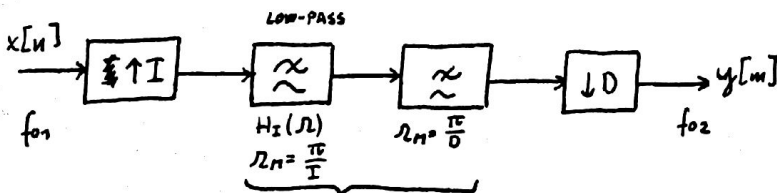
Ze imamo ne-celostevilsko razmerje med frekvencami, lahko uporabimo decimacijo + interpolacijo

### 7.4 Racionalen faktor spremembe vzorčne frekvence

$$\frac{f_{o2}}{f_{o1}} = k = \frac{I}{D}$$

$k > 1$  skupno interpolaciji  
 $k < 1$  skupno decimaciji

• najprej interpoliramo, nato decimiramo!



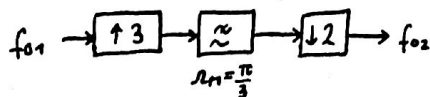
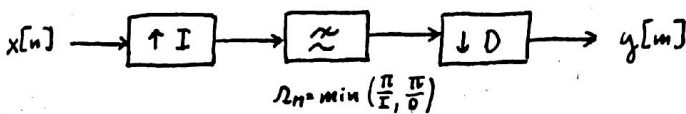
Zgled digitalni radio → DAT

$$f_{o1} = 32 \text{ kHz}$$

$$f_{o2} = 48 \text{ kHz}$$

$$f_{o2} = f_{o1} \cdot \frac{I}{D}$$

$$\frac{I}{D} = \frac{48}{32} = \frac{6}{4} = \frac{3}{2}$$

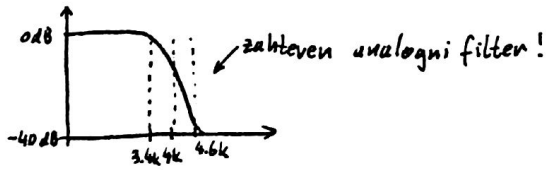


Zgled

Rekonstrukcija analognega signala z enostavnim nizkim filtrom:

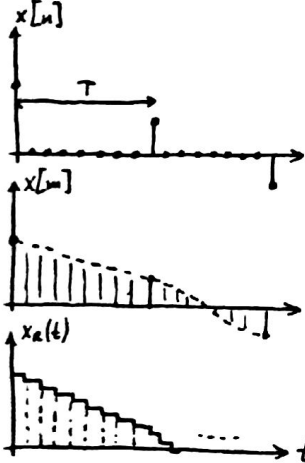
$f_z = 3400 \text{ Hz}$

$f_{o1} = 8 \text{ kHz}$



Rekonstrukcija z interpolacijo:

$I = 8$



$f_{o1} = 8 \text{ kHz} = \frac{1}{T}$

$f_{o2} = 64 \text{ kHz}$

$\tau' = T' = \frac{I}{8}$

$|H_R(\omega)| = \frac{\tau'}{T'} \cdot \text{sinc}\left(\frac{\omega\tau'}{2}\right) \sum_{k=-\infty}^{\infty} |X(\omega - k\omega_{o2})|$

$\tau' = \frac{1}{f_{o2}} = \frac{1}{2I f_z}$

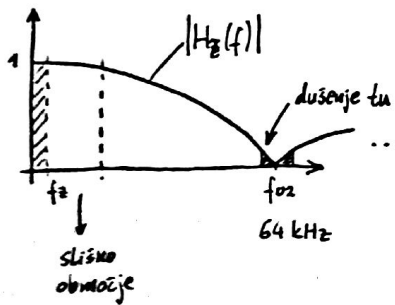
$|X_R(f)| = \text{sinc}(\pi \tau' f) \sum_{k=-\infty}^{\infty} |X(f - k f_{o2})|$

osnovni pas  $k=0$

$|X_R(f)| = \text{sinc}\left(\pi \frac{f_z}{f_{o2}}\right) \cdot |X(f)|$

vzamemo  $f = f_z$

$|X_R(f_z)| = \frac{\sin\left(\frac{\pi}{16}\right)}{\frac{\pi}{16}} |X(f_z)| = 0.993 |X(f_z)|$   
 $\downarrow$   
 $-0.05 \text{ dB}$



$|H_T(f)| = \frac{\sin\left(\pi \frac{f_{o2} - f_z}{f_{o2}}\right)}{\pi \frac{f_{o2} - f_z}{f_{o2}}} = \frac{\sin\left(\pi - \pi \frac{f_z}{f_{o2}}\right)}{\pi} = \frac{\sin\left(\pi \frac{f_z}{f_{o2}}\right)}{\pi} = \frac{f_z}{f_{o2}} = \frac{1}{16}$   
 $\hookrightarrow -24 \text{ dB}$

če dodamo še

RC sito

$f_p = \frac{1}{2\pi RC}$

$|H(f)|^2 = \frac{1}{1 + \left(\frac{f}{f_p}\right)^2}$

$-40 \text{ dB} = |H_{LP}(f)| + |H_T(f)|$

$|H_{LP}(f)| = -16 \text{ dB} \rightarrow \frac{f_{o2}}{f_p} = 6.22$

$f_p = 10 \text{ kHz}$

dušenje RC sita

pri  $f_z$

$|H(f)| = 0.88$   
 $f=f_z$

$\hookrightarrow -0.5 \text{ dB}$

## 8. Vpliv kvantizacije signala in koeficientov na lastnosti diskretnih sistemov

digitalna izvedba sistema povzroča dva efekta:

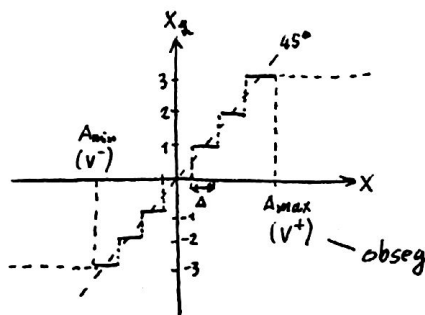
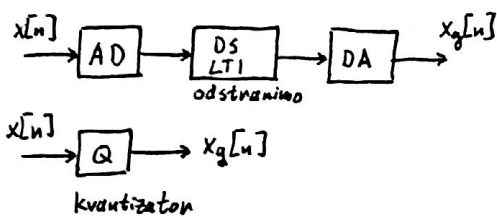
- kvantizacijski sum (napaka)
- odstopanje karakteristike nealiziranega sistema (odstopanje  $H(\Omega)$ )

Vznoki za degradacijo so:

- kvantizacija pri ADC
- zaokroževanje delnih rezultatov pri arit. operacijah

### 8.1. A/D Pretvorba signala

#### Kvantizacija



$$q[n] = x_q[n] - x[n]$$

↑  
kvantizacijska napaka (suma)

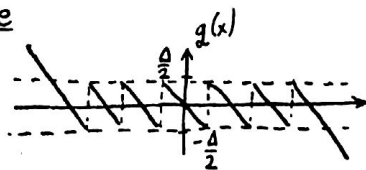
	koda z odmikom	dvojiški komplement
3	1 1 1	0 1 1
2	1 1 0	0 1 0
1	1 0 1	0 0 1
0	1 0 0	0 0 0
-1	0 1 1	1 1 1
-2	0 1 0	1 1 0
-3	0 0 1	1 0 1
-4	0 0 0	1 0 0

dve vrsti napake:

- zrnato popačenje (stopničke, vhod med  $v^-$  in  $v^+$ )
- prekomerno popačenje (vhod izven meja)

#### Moč zrnate napake

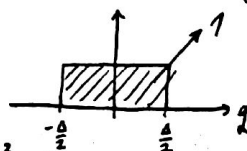
$$-\frac{\Delta}{2} \leq q_z \leq \frac{\Delta}{2}$$



porazdelitev verjetnosti za  $q_z$

$$p_{q_z}(q) = \begin{cases} \frac{1}{\Delta} & \text{za } -\frac{\Delta}{2} < q_z < \frac{\Delta}{2} \\ 0 & \text{drugod} \end{cases}$$

$$\begin{aligned} \sigma_{q_z}^2 &= \overline{q_z^2} = \int_{-\infty}^{\infty} q_z^2 p_{q_z}(q) dq = \\ &= \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} q^2 dq = \frac{1}{\Delta} \frac{q^3}{3} \Big|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} = \frac{\Delta^2}{12} = \overline{q_z^2} = \sigma_{q_z}^2 = N_z \end{aligned}$$



whaaaaa...?!

moč signala  $S = P_x = \sigma_x^2 = X_{ef}^2$

obseg kvantizatorja  $A_{min}$  do  $A_{max}$   
 (-A) (+A)

$A_{max} - A_{min} = 2A = M \cdot \Delta$   
 ↑  
 št. stopnic

$\Delta = \frac{2A}{M}$

$N_d = \frac{4A^2}{12\pi^2} = \frac{A^2}{3\pi^2}$

Razmerje med močjo signala in zrnatega popačenja:

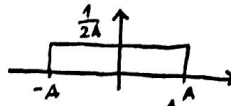
$SNR = \frac{S}{N_d} = \frac{X_{ef}^2}{\frac{A^2}{3\pi^2}} = \frac{3\pi^2 X_{ef}^2}{A^2} = M = 2^L \leftarrow \text{štev. bitov}$   
 $= \underline{\underline{3 \cdot 2^{2L} \left(\frac{X_{ef}}{A}\right)^2}}$

$SNR_{dB} = 10 \cdot \log SNR = 10 \cdot \log \left( 3 \cdot 2^{2L} \cdot \left(\frac{X_{ef}}{A}\right)^2 \right) = 10 \cdot \log 3 + (20 \cdot \log 2) \cdot L + 20 \cdot \log \left(\frac{X_{ef}}{A}\right)$

$SNR_{dB} = 4.77 \text{ dB} + 6.02 \text{ dB} \cdot L + \left(\frac{X_{ef}}{A}\right)_{dB}$  ← izkumuljuje

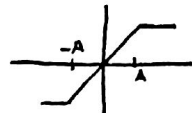
za sinusni signal:

za nek naključni signal:



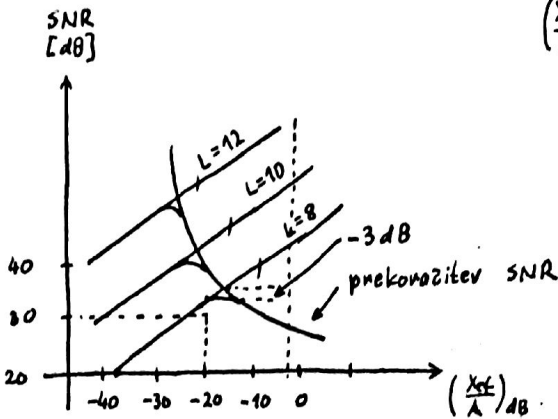
$X_{ef}^2 = \frac{1}{2A} \int_{-A}^A x^2 dx = \frac{A^2}{3}$

$\left(\frac{X_{ef}}{A}\right)_{dB} = 10 \cdot \log \frac{1}{3} = -4.77 \text{ dB}$

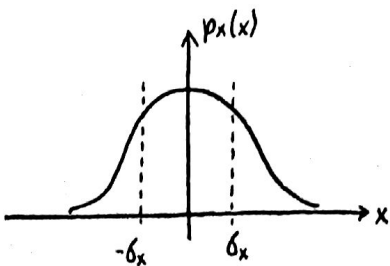


$X_{ef} = \frac{A}{\sqrt{2}}$   
 $\frac{X_{ef}}{A} = \frac{1}{\sqrt{2}}$

$\left(\frac{X_{ef}}{A}\right)_{dB} = -3 \text{ dB}$



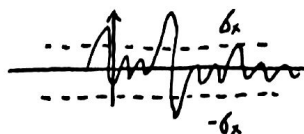
Moč prekoražitvene napake



$p_x(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x}{\sigma_x}\right)^2}$

signal napake

$q_p = \begin{cases} A-x & \text{za } x > A \\ -A-x & \text{za } x < -A \end{cases}$



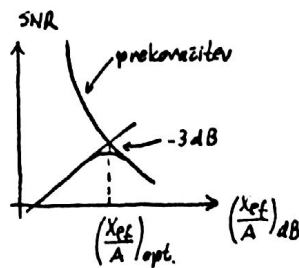
$\overline{Q_p^2} = \int_{-\infty}^A q_p^2 \cdot p_x(x) dx + \int_A^{\infty} q_p^2 \cdot p_x(x) dx$

zaradi simetrije

$N_p = \overline{Q_p^2} = 2 \cdot \int_A^{\infty} q_p^2 \cdot p_x(x) dx = 2 \int_A^{\infty} (A-x)^2 \cdot p_x(x) dx$

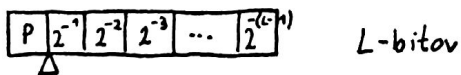
za Gaussovo porazdelitev velja

$$\left(\frac{X_{eff}}{A}\right)_{dB} = -12 \text{ dB} \rightarrow \frac{S}{N_p} = 50 \text{ dB}$$



Napaka pri zaokroževanju vmesnih ete rezultatov:

Zapis s stalno decimalno vejico



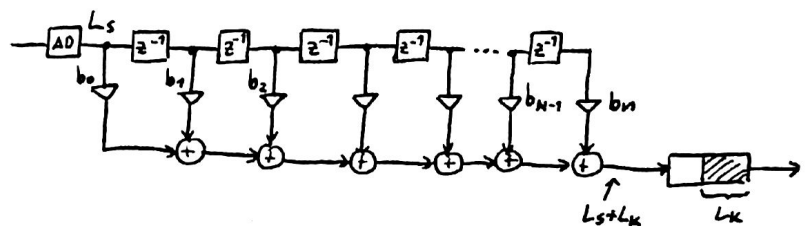
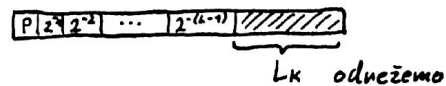
signal je zapisan z  $L_s$ -biti  
koeficienti pa z  $L_k$ -biti

za produkt rabimo  $L_s + L_k - 1$  bitov

za FIR sisteme velja

$$y[n] = \sum_{m=0}^N b_m x[n-m]$$

a) vmesni rezultati so shranjeni z  $L_s + L_k$  ~~X~~ biti



b) vmesni rezultati so shranjeni z  $L_s$  biti

vsako zaokroževanje generira nek "šum"

$$N_{zish} = \underbrace{(N+1)}_{\text{zaokroževanje}} \cdot N_2 + \underbrace{N_2}_{\text{AD pretvorba}}$$

za večji ned  
 $N_{zish} = N \cdot N_2$   
↑  
red

$$\underline{\underline{SNR_{zish} = \frac{S}{2 N_{gz}} = \text{poslabšanje za 3 dB}}}$$

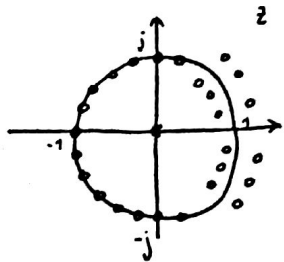
$$SNR_{zish} = 10 \cdot \log \frac{S}{N \cdot N_2} = 10 \cdot \log \frac{S}{N_2} - 10 \cdot \log N$$

$$\underline{\underline{SNR_{zish} = SNR_{(dB)} - 10 \cdot \log N}}$$

Pri IIR sitih si zaokroževanja ne smemo privoščiti ...

8.2. Kvantizacija koeficientov

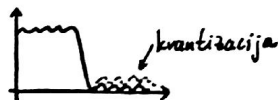
FIR sita (ničle določajo potek)



- kvantizacija ne pokvari linearne faze!  
simetrična  $h[n]$  se ohlani  $b_0=b_N$

- v prepustnem pasu  $H(\Omega)$  določajo ničle so odmaknjene od krožnice, majhni premiki majhen efekt

- odmik ničle od krožnice v zapori se bolj pozna  
dušenje v zapori se poslabša

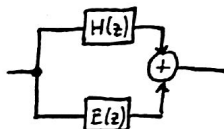


Model za analizo odstopanj

$$H(z) = \sum_{n=0}^N h[n] z^{-n}$$

$$\hat{H}(z) = \sum_{n=0}^N (h[n] + e[n]) z^{-n} = H(z) + E(z)$$

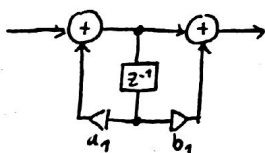
$$E(z) = \sum_{n=0}^N e[n] z^{-n}$$



IIR sita (poli določajo potek)

Zaradi polov v bližini krožnice, so občutljivi na kvantizacijo  
direktna struktura višjih redov ( $N > 2$ ) so neuporabne  
uporabljamo kaskado celic drugega reda (in prvega)

Sistem 1. reda



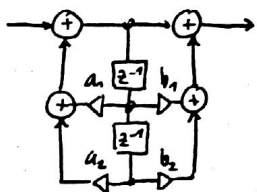
$$H(z) = \frac{1 + b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{z + b_1}{z - a_1}$$

$$p_1 = a_1$$

$$n_1 = -b_1$$

obseg koeficientov  $-1 < a_1 < 1$

Sistem 2. reda



$$L_k = 8 \text{ ločljivost } \frac{1}{128}$$

$$H(z) = \frac{1 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2}$$

poli  $p_1 = r \cdot e^{j\Omega_p}$   
 $p_2 = r \cdot e^{-j\Omega_p}$

$$(z - p_1)(z - p_2) = (z - r e^{j\Omega_p})(z - r e^{-j\Omega_p}) =$$

$$-a_2 = r^2$$

$$n = |p| = \sqrt{Q(-a_2)}$$

$$a_1 = 2r \cos \Omega_p = 2 \operatorname{Re} \{p\}$$

$$\operatorname{Re} \{p\} = \frac{Q(a_1)}{2}$$

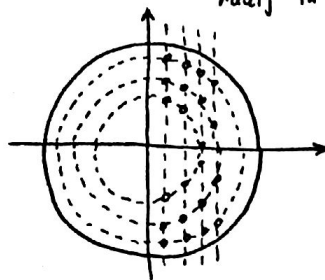
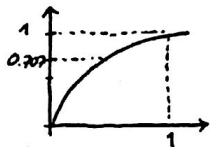
kvantizacijska vrednost

Koeficient  $-2 < a_1 < 2$

kvantizacija realne komponente pola  $\Delta = \frac{4}{2^L \cdot 2} = \frac{4}{2^{L+1}} = 2^{-(L+1)}$

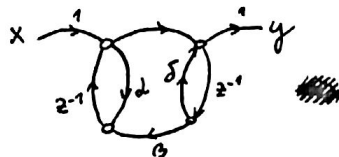
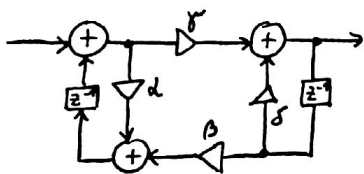
radij:

radij in realni del pola



nize slabše vplivajo na frekvenčni odziv, zato je tu manj problemov ( $b_2$  koeficienta ne prizadaneemo, ker je 1)

Sklopljena struktura  
(coupled form)



$$y = ax + b \cdot y$$

$$(1-b)y = ax$$

$$y = \frac{a}{1-b} x$$

$$x \xrightarrow{\frac{a}{1-b}} y$$

$$H(z) = \frac{\gamma \cdot z^2}{z^2 - (d+\delta)z + (d\delta - \beta\gamma)}$$

izberemo

$$d = \delta$$

$$\gamma = -\beta$$

$$p_1 = p_2 = d \pm j\beta$$

dobimo enakomerno mrežo

$$d^2 + \beta^2 = |p|^2$$

poenostavitev

